Supplement to "Two-step estimation for time varying ARCH models"

Yuanyuan Zhang[a], Rong Liu[b], Qin Shao[b] and Lijian Yang[a]
[a] Tsinghua University, China and [b] University of Toledo, USA

Proof of Lemma A.1. Note that

$$\frac{\widehat{Z}_t - Z_t}{Y_{t,n}^2} - \frac{g(t/n) - \hat{g}(t/n)}{g^2(t/n)} = \frac{g(t/n) - \hat{g}(t/n)}{\hat{g}(t/n)\, g(t/n)} - \frac{g(t/n) - \hat{g}(t/n)}{g^2(t/n)}.$$

Hence (A.1) and $\hat{g}(t/n) \geq g(t/n) - \Delta_{n,0} \geq c_l - \Delta_{n,0}$ together entail that

$$\begin{aligned}\Delta_{n,1} &= \max_{p+1\leq t\leq n} \left| \frac{\{g(t/n) - \hat{g}(t/n)\}^2}{g^2(t/n)\, \hat{g}(t/n)} \right| \\ &\leq (c_l - \Delta_{n,0})^{-1} c_l^{-2} \Delta_{n,0}^2 = \mathcal{O}_p\left(\Delta_{n,0}^2\right).\end{aligned}$$

From (2.9) and (A.2),

$$\begin{aligned}\Delta_{n,2} &\leq \max_{p+1\leq t\leq n} \left| \frac{\widehat{Z}_t - Z_t}{Y_{t,n}^2} - \frac{g(t/n) - \hat{g}(t/n)}{g^2(t/n)} \right| \\ &\quad + \max_{p+1\leq t\leq n} \left| \frac{g(t/n) - \hat{g}(t/n)}{g^2(t/n)} \right| \\ &\leq \Delta_{n,1} + c_l^{-2} \Delta_{n,0} = \mathcal{O}_p(\Delta_{n,0}).\end{aligned}$$

Thus (A.4) and (A.5) hold.

At last, for $1 \leq k \leq p$ and $p+1 \leq t \leq n$,

$$\begin{aligned}\left| \frac{\hat{g}((t-k)/n)}{g((t-k)/n)} - \frac{\hat{g}(t/n)}{g(t/n)} \right| &\leq \left| \frac{\{\hat{g}((t-k)/n) - \hat{g}(t/n)\}}{g((t-k)/n)} \right| \\ &\quad + \left| \frac{\{g((t-k)/n) - g(t/n)\}\, \hat{g}(t/n)}{g((t-k)/n)\, g(t/n)} \right| \\ &\leq c_l^{-1} \|\hat{g}\|_{0,r} \left(\frac{k}{n}\right)^r + c_l^{-2} \|g\|_{0,r} (c_u + \Delta_{n,0}) \left(\frac{k}{n}\right)^r.\end{aligned}$$

So (A.6) is proved and also the lemma. □

Lemma A.2. *Under Assumptions (a)-(c), for any fixed $1 \le k \le p$,*

$$T_{n,1} = (n-p)^{-1} \sum_{t=p+1}^{n} Z_{t-k} \left\{ \widehat{Z}_t - Z_t - \sum_{k'=1}^{p} \alpha_{0,k'} \left(\widehat{Z}_{t-k'} - Z_{t-k'} \right) \right\} = o_p\left(n^{-1/2}\right), \tag{A.10}$$

$$T_{n,2} = (n-p)^{-1} \sum_{t=p+1}^{n} \left(\widehat{Z}_{t-k} - Z_{t-k} \right) \xi_t = o_p\left(n^{-1/2}\right), \tag{A.11}$$

$$T_{n,3} = (n-p)^{-1} \sum_{t=p+1}^{n} \left(\widehat{Z}_{t-k} - Z_{t-k} \right) \times \left\{ \widehat{Z}_t - Z_t - \sum_{k'=1}^{p} \alpha_{0,k'} \left(\widehat{Z}_{t-k'} - Z_{t-k'} \right) \right\} = o_p\left(n^{-1/2}\right). \tag{A.12}$$

Proof. Making use of $Y_{t,n}^2 = (Z_t + 1)\, g(t/n)$, one first decomposes the term $T_{n,1}$ as follows:

$$T_{n,1} = T_{n,1,1} + T_{n,1,2} + T_{n,1,3},$$

where

$$\begin{aligned} T_{n,1,1} &= (n-p)^{-1} \sum_{t=p+1}^{n} Z_{t-k} \left\{ \widehat{Z}_t - Z_t - Y_{t,n}^2 \frac{g(t/n) - \hat{g}(t/n)}{g^2(t/n)} \right\} \\ &\quad - (n-p)^{-1} \sum_{t=p+1}^{n} Z_{t-k} \\ &\quad \times \left[\sum_{k'=1}^{p} \alpha_{0,k'} \left\{ \widehat{Z}_{t-k'} - Z_{t-k'} - Y_{t-k',n}^2 \frac{g((t-k')/n) - \hat{g}((t-k')/n)}{g^2((t-k')/n)} \right\} \right], \end{aligned}$$

$$\begin{aligned} T_{n,1,2} &= (n-p)^{-1} \sum_{t=p+1}^{n} Z_{t-k} \\ &\quad \times \sum_{k'=1}^{p} \alpha_{0,k'} (Z_{t-k'} + 1) \left\{ \frac{g(t/n) - \hat{g}(t/n)}{g(t/n)} - \frac{g((t-k')/n) - \hat{g}((t-k')/n)}{g((t-k')/n)} \right\}, \end{aligned}$$

$$T_{n,1,3} = (n-p)^{-1} \sum_{t=p+1}^{n} Z_{t-k} \left\{ (Z_t + 1) - \sum_{k'=1}^{p} \alpha_{0,k'} (Z_{t-k'} + 1) \right\} \frac{g(t/n) - \hat{g}(t/n)}{g(t/n)}.$$

Notice that by the definition of $\Delta_{n,1}$ in (A.1)

$$\left| \widehat{Z}_t - Z_t - Y_{t,n}^2 \frac{g(t/n) - \hat{g}(t/n)}{g^2(t/n)} \right| \le \Delta_{n,1} Y_{t,n}^2, p+1 \le t \le n.$$

Thus

$$\begin{aligned}\left|T_{n,1,1}\right| &\leq \Delta_{n,1} \times (n-p)^{-1} \sum_{t=p+1}^{n} \left|Z_{t-k}\right| \left(Y_{t,n}^2 + \sum_{k'=1}^{p} \left|\alpha_{0,k'}\right| Y_{t-k',n}^2\right) \\ &\leq \Delta_{n,1} c_u \times (n-p)^{-1} \sum_{t=p+1}^{n} \left|Z_{t-k}\right| \left(X_t^2 + \sum_{k'=1}^{p} \left|\alpha_{0,k'}\right| X_{t-k'}^2\right) \\ &= \mathcal{O}_p\left(\Delta_{n,1}\right),\end{aligned}$$

as
$(n-p)^{-1} \sum_{t=p+1}^{n} \left|Z_{t-k}\right| \left(X_t^2 + \sum_{k'=1}^{p} \left|\alpha_{0,k'}\right| X_{t-k'}^2\right) \overset{a.s.}{\to} \mathsf{E}\left|Z_{t-k}\right| \left(X_t^2 + \sum_{k'=1}^{p} \left|\alpha_{0,k'}\right| X_{t-k'}^2\right)$ by Theorem A.2 of Francq and Zakoian (2010) for the strictly stationary time series $\left|Z_{t-k}\right| \left(X_t^2 + \sum_{k'=1}^{p} \left|\alpha_{0,k'}\right| X_{t-k'}^2\right)$ with a finite mean.

Next, note that by the definition of $\Delta_{n,3}$ in (A.3) and its property in (A.6)

$$\begin{aligned}\left|T_{n,1,2}\right| &\leq (n-p)^{-1} \sum_{t=p+1}^{n} \left|Z_{t-k}\right| \sum_{k'=1}^{p} \left|\alpha_{0,k'}\right| \left(\left|Z_{t-k'}\right| + 1\right) \Delta_{n,3} = \mathcal{O}_p\left(\Delta_{n,3}\right) \\ &= \mathcal{O}_p\left(n^{-r}\right) = o_p\left(n^{-1/2}\right),\end{aligned}$$

as $(n-p)^{-1} \sum_{t=p+1}^{n} \left|Z_{t-k}\right| \sum_{k'=1}^{p} \left|\alpha_{0,k'}\right| \left(\left|Z_{t-k'}\right| + 1\right) \overset{a.s.}{\to} \mathsf{E}\left|Z_{t-k}\right| \sum_{k'=1}^{p} \left|\alpha_{0,k'}\right| \left(\left|Z_{t-k'}\right| + 1\right)$ by Theorem A.2 of Francq and Zakoian (2010) for the strictly stationary time series $\left|Z_{t-k}\right| \sum_{k'=1}^{p} \left|\alpha_{0,k'}\right| \left(\left|Z_{t-k'}\right| + 1\right)$ with a finite mean.

Finally, according to (2.12) in Assumption (c), with $\zeta_t = Z_{t-k}$ or $Z_{t-k}\xi_t$,

$$\begin{aligned}T_{n,1,3} &= (n-p)^{-1} \sum_{t=p+1}^{n} Z_{t-k} \left\{\left(Z_t + 1\right) - \sum_{k'=1}^{p} \alpha_{0,k'} \left(Z_{t-k'} + 1\right)\right\} \frac{g(t/n) - \hat{g}(t/n)}{g(t/n)} \\ &= (n-p)^{-1} \sum_{t=p+1}^{n} \left\{Z_{t-k}\xi_t + \left(1 - \sum_{k'=1}^{p} \alpha_{0,k'}\right) Z_{t-k}\right\} \frac{g(t/n) - \hat{g}(t/n)}{g(t/n)} \\ &= o_p\left(n^{-1/2}\right).\end{aligned}$$

Putting together the above bounds on $T_{n,1,1}, T_{n,1,2}$ and $T_{n,1,3}$, (A.10) is proved. Similarly, (A.11) and (A.12) can be proved. □

A.1 Proof of Theorem 2.1

Recall that for $1 \leq k, k' \leq p$

$$\begin{aligned}
\widetilde{\gamma}(k, k') &= (n-p)^{-1} \sum_{t=p+1}^{n} Z_{t-k} Z_{t-k'}, \\
\widehat{\gamma}(k, k') &= (n-p)^{-1} \sum_{t=p+1}^{n} \widehat{Z}_{t-k} \widehat{Z}_{t-k'}.
\end{aligned} \tag{A.13}$$

Define another measure of deviation for the sample autocovariance

$$\Delta_{n,4} = \Delta_{n,4}(\hat{g}) = \max_{1 \leq k_1, k_2 \leq p} \left| \widehat{\gamma}(k_1, k_2) - \widetilde{\gamma}(k_1, k_2) \right|.$$

The next lemma shows that $\Delta_{n,4}$ converges to 0 at the same rate as $\Delta_{n,0}$.

Lemma A.3. *Under Assumptions (a)-(c), as $n \to \infty$, for $1 \leq k_1, k_2 \leq p$,*

$$\begin{aligned}
&\widehat{\gamma}(k_1, k_2) - \widetilde{\gamma}(k_1, k_2) \\
= \ & 2(n-p)^{-1} \sum_{t=p+1}^{n} Z_{t-k_1} Z_{t-k_2} \frac{g(t/n) - \hat{g}(t/n)}{g(t/n)} + o_p\left(n^{-1/2}\right),
\end{aligned}$$

$$\Delta_{n,4} = \mathcal{O}_p(\Delta_{n,0}) = o_p\left(n^{-1/4}\right). \tag{A.14}$$

Proof. Consider

$$\begin{aligned}
&\widehat{\gamma}(k_1, k_2) - \widetilde{\gamma}(k_1, k_2) \\
= \ & (n-p)^{-1} \sum_{t=p+1}^{n} \left(\widehat{Z}_{t-k_1} \widehat{Z}_{t-k_2} - Z_{t-k_1} Z_{t-k_2} \right) \\
= \ & (n-p)^{-1} \sum_{t=p+1}^{n} \left(\widehat{Z}_{t-k_1} - Z_{t-k_1} \right) Z_{t-k_2} \\
& + (n-p)^{-1} \sum_{t=p+1}^{n} \left(\widehat{Z}_{t-k_2} - Z_{t-k_2} \right) Z_{t-k_1} \\
& + (n-p)^{-1} \sum_{t=p+1}^{n} \left(\widehat{Z}_{t-k_2} - Z_{t-k_2} \right) \left(\widehat{Z}_{t-k_1} - Z_{t-k_1} \right) \\
= \ & (n-p)^{-1} \sum_{t=p+1}^{n} \left(\widehat{Z}_{t-k_1} - Z_{t-k_1} \right) Z_{t-k_2} \\
& + (n-p)^{-1} \sum_{t=p+1}^{n} \left(\widehat{Z}_{t-k_2} - Z_{t-k_2} \right) Z_{t-k_1} + \mathcal{O}_p\left(\Delta_{n,0}^2\right).
\end{aligned}$$

Note that

$$
\begin{aligned}
& (n-p)^{-1} \sum_{t=p+1}^{n} \left(\widehat{Z}_{t-k_1} - Z_{t-k_1}\right) Z_{t-k_2} \\
= \; & (n-p)^{-1} \sum_{t=p+1}^{n} Y_{t-k_1,n}^2 \frac{g\left(\left(t-k_1\right)/n\right) - \hat{g}\left(\left(t-k_1\right)/n\right)}{g^2\left(\left(t-k_1\right)/n\right)} Z_{t-k_2} + \mathcal{O}_p\left(\Delta_{n,0}^2\right) \\
= \; & (n-p)^{-1} \sum_{t=p+1}^{n} \left(Z_{t-k_1} + 1\right) \frac{g\left(\left(t-k_1\right)/n\right) - \hat{g}\left(\left(t-k_1\right)/n\right)}{g\left(\left(t-k_1\right)/n\right)} Z_{t-k_2} + \mathcal{O}_p\left(\Delta_{n,0}^2\right) \\
= \; & (n-p)^{-1} \sum_{t=p+1}^{n} Z_{t-k_1} Z_{t-k_2} \frac{g\left(t/n\right) - \hat{g}\left(t/n\right)}{g\left(t/n\right)} + o_p\left(n^{-1/2}\right).
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
& (n-p)^{-1} \sum_{t=p+1}^{n} \left(\widehat{Z}_{t-k_2} - Z_{t-k_2}\right) Z_{t-k_1} \\
= \; & (n-p)^{-1} \sum_{t=p+1}^{n} Z_{t-k_1} Z_{t-k_2} \frac{g\left(t/n\right) - \hat{g}\left(t/n\right)}{g\left(t/n\right)} + o_p\left(n^{-1/2}\right).
\end{aligned}
$$

Thus (A.14) is proved by noting that

$$
\begin{aligned}
& \left| (n-p)^{-1} \sum_{t=p+1}^{n} Z_{t-k_1} Z_{t-k_2} \frac{g\left(t/n\right) - \hat{g}\left(t/n\right)}{g\left(t/n\right)} \right| \\
\leq \; & c_l^{-1} \Delta_{n,0} \times (n-p)^{-1} \sum_{t=p+1}^{n} \left| Z_{t-k_1} Z_{t-k_2} \right|,
\end{aligned}
$$

and that $(n-p)^{-1} \sum_{t=p+1}^{n} \left|Z_{t-k_1} Z_{t-k_2}\right| \xrightarrow{a.s.} \mathsf{E}\left|Z_{t-k_1} Z_{t-k_2}\right|$ by Theorem A.2 of Francq and Zakoian (2010) for the strictly stationary time series $\left|Z_{t-k_1} Z_{t-k_2}\right|$ with a finite mean.
□

Proof of Theorem 2.1. Notice that the k-th element $(1 \leq k \leq p)$ of $\widehat{\boldsymbol{\gamma}} - \widetilde{\boldsymbol{\gamma}} -$

$\left(\widehat{\mathbf{\Gamma}}-\widetilde{\mathbf{\Gamma}}\right)\boldsymbol{\alpha}_0$ is decomposed as

$$\begin{aligned}
& n^{-1}\sum_{t=p+1}^{n}\left(\widehat{Z}_{t-k}\widehat{Z}_t - Z_{t-k}Z_t\right) - n^{-1}\sum_{t=p+1}^{n}\sum_{k'=1}^{p}\alpha_{0,k'}\left(\widehat{Z}_{t-k}\widehat{Z}_{t-k'} - Z_{t-k}Z_{t-k'}\right) \\
= \; & n^{-1}\sum_{t=p+1}^{n}\widehat{Z}_{t-k}\left(\widehat{Z}_t - \sum_{k'=1}^{p}\alpha_{0,k'}\widehat{Z}_{t-k'}\right) \\
& -n^{-1}\sum_{t=p+1}^{n}Z_{t-k}\left(Z_t - \sum_{k'=1}^{p}\alpha_{0,\nu'}Z_{t-k'}\right) \\
= \; & n^{-1}\sum_{t=p+1}^{n}Z_{t-k}\left(\widehat{Z}_t - \sum_{k'=1}^{p}\alpha_{0,k'}\widehat{Z}_{t-k'} - Z_t + \sum_{k'=1}^{p}\alpha_{0,k'}Z_{t-k'}\right) \\
& +n^{-1}\sum_{t=p+1}^{n}\left(\widehat{Z}_{t-k} - Z_{t-k}\right)\left(Z_t - \sum_{k'=1}^{p}\alpha_{0,k'}Z_{t-k'}\right) \\
& +n^{-1}\sum_{t=p+1}^{n}\left(\widehat{Z}_{t-k} - Z_{t-k}\right)\left(\widehat{Z}_t - \sum_{k'=1}^{p}\alpha_{0,k'}\widehat{Z}_{t-k'} - Z_t + \sum_{k'=1}^{p}\alpha_{0,k'}Z_{t-k'}\right) \\
= \; & T_{n,1} + T_{n,2} + T_{n,3} = o_p\left(n^{-1/2}\right),
\end{aligned}$$

where $\{T_{n,i}\}_{i=1}^{3}$ are defined in (A.10)-(A.12). The order above is obtained by Lemma A.2. Hence,

$$\widehat{\boldsymbol{\gamma}} - \widetilde{\boldsymbol{\gamma}} - \left(\widehat{\mathbf{\Gamma}}-\widetilde{\mathbf{\Gamma}}\right)\boldsymbol{\alpha}_0 = o_p\left(n^{-1/2}\right).$$

Next, since $\widehat{\mathbf{\Gamma}}\widehat{\boldsymbol{\alpha}}_{\mathrm{LSE}} = \widehat{\boldsymbol{\gamma}}$ and $\widetilde{\mathbf{\Gamma}}\widetilde{\boldsymbol{\alpha}}_{\mathrm{LSE}} = \widetilde{\boldsymbol{\gamma}}$,

$$\widehat{\mathbf{\Gamma}}\widehat{\boldsymbol{\alpha}}_{\mathrm{LSE}} - \widetilde{\mathbf{\Gamma}}\widetilde{\boldsymbol{\alpha}}_{\mathrm{LSE}} - \left(\widehat{\mathbf{\Gamma}}-\widetilde{\mathbf{\Gamma}}\right)\boldsymbol{\alpha}_0 = o_p\left(n^{-1/2}\right).$$

Hence

$$\widehat{\mathbf{\Gamma}}\left(\widehat{\boldsymbol{\alpha}}_{\mathrm{LSE}} - \widetilde{\boldsymbol{\alpha}}_{\mathrm{LSE}}\right) + \left(\widehat{\mathbf{\Gamma}}-\widetilde{\mathbf{\Gamma}}\right)\left(\widetilde{\boldsymbol{\alpha}}_{\mathrm{LSE}} - \boldsymbol{\alpha}_0\right) = o_p\left(n^{-1/2}\right).$$

According to Proposition 1, $\widetilde{\boldsymbol{\alpha}}_{\mathrm{LSE}} - \boldsymbol{\alpha}_0 = \mathcal{O}_p\left(n^{-1/2}\right)$, which together with $\widehat{\mathbf{\Gamma}}-\widetilde{\mathbf{\Gamma}} = \mathcal{O}_p\left(\Delta_{n,0}\right)$ from (A.14) implies that

$$\left(\widehat{\mathbf{\Gamma}}-\widetilde{\mathbf{\Gamma}}\right)\left(\widetilde{\boldsymbol{\alpha}}_{\mathrm{LSE}} - \boldsymbol{\alpha}_0\right) = \mathcal{O}_p\left(n^{-1/2}\times\Delta_{n,0}\right) = o_p\left(n^{-1/2}\right).$$

Thus

$$\widehat{\mathbf{\Gamma}}\left(\widehat{\boldsymbol{\alpha}}_{\mathrm{LSE}} - \widetilde{\boldsymbol{\alpha}}_{\mathrm{LSE}}\right) = o_p\left(n^{-1/2}\right).$$

Finally, $\widehat{\mathbf{\Gamma}}-\widetilde{\mathbf{\Gamma}} = \mathcal{O}_p\left(\Delta_{n,0}\right)$ and $\widetilde{\mathbf{\Gamma}} \overset{P}{\to} \mathbf{\Gamma}$ with $\mathbf{\Gamma}$ defined in (2.4). Slutsky's Theorem implies that $\widehat{\mathbf{\Gamma}} \overset{P}{\to} \mathbf{\Gamma}$, a positive definite matrix. Therefore

$$\widehat{\boldsymbol{\alpha}}_{\mathrm{LSE}} - \widetilde{\boldsymbol{\alpha}}_{\mathrm{LSE}} = o_p\left(n^{-1/2}\right).$$

by Slutsky's Theorem, and the proof is complete. □

A.2 Proof of Theorem 2.2

According to (2.7), $\widehat{X}_t^2 = \widehat{Z}_t + 1$,

$$\widehat{X}_t^2 - X_t^2 = \widehat{Z}_t - Z_t = \frac{\widehat{Z}_t - Z_t}{Y_{t,n}^2} g\left(t/n\right)\left(Z_t + 1\right)$$

with (A.2) implies that

$$\left|\widehat{X}_t^2 - X_t^2\right| = \left|\frac{\widehat{Z}_t - Z_t}{Y_{t,n}^2}\right| g\left(t/n\right)\left|Z_t + 1\right| \leq c_u \Delta_{n,2}\left|Z_t + 1\right|, \tag{A.15}$$

while $\widehat{\sigma}_t^2\left(\boldsymbol{\alpha}\right) - \sigma_t^2\left(\boldsymbol{\alpha}\right) = \sum_{k=1}^p \alpha_k \left(\widehat{X}_{t-k}^2 - X_{t-k}^2\right)$ entails that

$$\left|\widehat{\sigma}_t^2\left(\boldsymbol{\alpha}\right) - \sigma_t^2\left(\boldsymbol{\alpha}\right)\right| \leq c_u \Delta_{n,2} \sum_{k=1}^p \left|\alpha_k\right|\left|Z_{t-k} + 1\right|,$$

$$\left|\log \widehat{\sigma}_t^2\left(\boldsymbol{\alpha}\right) - \log \sigma_t^2\left(\boldsymbol{\alpha}\right)\right| \leq \left(1 - \sum_{k=1}^p \alpha_k\right)^{-1} c_u \Delta_{n,2} \sum_{k=1}^p \left|\alpha_k\right|\left|Z_{t-k} + 1\right|. \tag{A.16}$$

Lemma A.4. *Under Assumptions (a), (b) and (d),*

$$(n-p)^{-1} \sup_{\boldsymbol{\alpha} \in \boldsymbol{\Xi}} \left|\sum_{t=p+1}^n \left\{\log \widehat{\sigma}_t^2\left(\boldsymbol{\alpha}\right) - \log \sigma_t^2\left(\boldsymbol{\alpha}\right)\right\}\right| = \mathcal{O}_p\left(\Delta_{n,0}\right) = o_p\left(n^{-1/4}\right), \tag{A.17}$$

$$(n-p)^{-1} \sup_{\boldsymbol{\alpha} \in \boldsymbol{\Xi}} \left|\sum_{t=p+1}^n \left\{\frac{\widehat{X}_t^2}{\widehat{\sigma}_t^2\left(\boldsymbol{\alpha}\right)} - \frac{X_t^2}{\sigma_t^2\left(\boldsymbol{\alpha}\right)}\right\}\right| = \mathcal{O}_p(\Delta_{n,0}) = o_p\left(n^{-1/4}\right).$$

Proof. Applying (A.15) and (A.16), and the fact that $\sup_{\boldsymbol{\alpha} \in \boldsymbol{\Xi}} \left(1 - \sum_{k=1}^p \alpha_k\right)^{-1} < \infty$ according to Assumption (a), one argues that

$$(n-p)^{-1} \sum_{t=p+1}^n \left(\widehat{X}_t^2 - X_t^2\right) \leq c_u \Delta_{n,2} \left(n-p\right)^{-1} \sum_{t=p+1}^n \left|Z_t + 1\right| = \mathcal{O}_p\left(\Delta_{n,0}\right), \tag{A.18}$$

$$\begin{aligned}
&(n-p)^{-1} \sup_{\boldsymbol{\alpha} \in \boldsymbol{\Xi}} \left|\sum_{t=p+1}^n \left\{\log \sigma_t^2(\boldsymbol{\alpha}) - \log \widehat{\sigma}_t^2(\boldsymbol{\alpha})\right\}\right| \\
&\leq c_u \Delta_{n,2} \sup_{\boldsymbol{\alpha} \in \boldsymbol{\Xi}} \left(1 - \sum_{k=1}^p \alpha_k\right)^{-1} (n-p)^{-1} \sum_{t=p+1}^n \sum_{k=1}^p \left|\alpha_k\right|\left|Z_{t-k} + 1\right| = \mathcal{O}_p\left(\Delta_{n,0}\right).
\end{aligned}$$

The order in the last step makes use of $\min\left(\sigma_t^2(\boldsymbol{\alpha}), \widehat{\sigma}_t^2(\boldsymbol{\alpha})\right) \geq 1 - \sum_{k=1}^{p} \alpha_k$. Furthermore,

$$
\begin{aligned}
& (n-p)^{-1} \sup_{\boldsymbol{\alpha}\in\boldsymbol{\Xi}} \left| \sum_{t=p+1}^{n} \left\{ \frac{\widehat{X}_t^2}{\widehat{\sigma}_t^2(\boldsymbol{\alpha})} - \frac{X_t^2}{\sigma_t^2(\boldsymbol{\alpha})} \right\} \right| \\
= \; & (n-p)^{-1} \sup_{\boldsymbol{\alpha}\in\boldsymbol{\Xi}} \left| \sum_{t=p+1}^{n} \left[\frac{\widehat{X}_t^2 - X_t^2}{\widehat{\sigma}_t^2(\boldsymbol{\alpha})} + \frac{X_t^2\left\{\sigma_t^2(\boldsymbol{\alpha}) - \widehat{\sigma}_t^2(\boldsymbol{\alpha})\right\}}{\widehat{\sigma}_t^2(\boldsymbol{\alpha})\,\sigma_t^2(\boldsymbol{\alpha})} \right] \right| \\
\leq \; & \sup_{\boldsymbol{\alpha}\in\boldsymbol{\Xi}} \left(1 - \sum_{k=1}^{p} \alpha_k\right)^{-1} (n-p)^{-1} \sum_{t=p+1}^{n} \left| \widehat{X}_t^2 - X_t^2 \right| \\
& + \sup_{\boldsymbol{\alpha}\in\boldsymbol{\Xi}} \left(1 - \sum_{k=1}^{p} \alpha_k\right)^{-2} (n-p)^{-1} \sum_{t=p+1}^{n} \left|X_t^2\right| \left|\sigma_t^2(\boldsymbol{\alpha}) - \widehat{\sigma}_t^2(\boldsymbol{\alpha})\right| \\
= \; & O_p(\Delta_{n,0}).
\end{aligned}
$$

The proof is complete. □

Corollary A.1. *Under Assumptions (a), (b) and (d),*

$$
\sup_{\boldsymbol{\alpha}\in\boldsymbol{\Xi}} \left| \widehat{Q}_n(\boldsymbol{\alpha}) - Q_n(\boldsymbol{\alpha}) \right| = \mathcal{O}_p(\Delta_{n,0}) = o_p\left(n^{-1/4}\right),
$$

and consequently $\sup_{\boldsymbol{\alpha}\in\boldsymbol{\Xi}} \left| \widehat{Q}_n(\boldsymbol{\alpha}) - \mathsf{E}Q_n(\boldsymbol{\alpha}) \right| = o_p(1)$, *and* $\widehat{\boldsymbol{\alpha}}_{\text{MLE}} \xrightarrow{P} \boldsymbol{\alpha}_0$.

Note that for any $1 \leq k \leq p$,

$$
\frac{\partial Q_n(\boldsymbol{\alpha})}{\partial \alpha_k} = (n-p)^{-1} \sum_{t=p+1}^{n} \frac{X_{t-k}^2 - 1}{\sigma_t^2(\boldsymbol{\alpha})} - (n-p)^{-1} \sum_{t=p+1}^{n} \frac{X_t^2}{\sigma_t^4(\boldsymbol{\alpha})} \left(X_{t-k}^2 - 1\right)
$$

and hence

$$
\frac{\partial Q_n(\boldsymbol{\alpha})}{\partial \alpha_k} = -(n-p)^{-1} \sum_{t=p+1}^{n} \frac{Z_{t-k}\left(\sum_{l=1}^{p} \alpha_l Z_{t-l} - Z_t\right)}{\sigma_t^4(\boldsymbol{\alpha})}, \tag{A.19}
$$

while for any $1 \leq k, l \leq p$,

$$
\frac{\partial^2 Q_n(\boldsymbol{\alpha})}{\partial \alpha_k \partial \alpha_l} = -(n-p)^{-1} \sum_{t=p+1}^{n} \frac{Z_{t-k} Z_{t-l}}{\sigma_t^4(\boldsymbol{\alpha})} + 2(n-p)^{-1} \sum_{t=p+1}^{n} \frac{Z_{t-k} Z_{t-l} X_t^2}{\sigma_t^6(\boldsymbol{\alpha})}. \tag{A.20}
$$

Likewise

$$
\frac{\partial \widehat{Q}_n(\boldsymbol{\alpha})}{\partial \alpha_k} = -(n-p)^{-1} \sum_{t=p+1}^{n} \frac{\widehat{Z}_{t-k}\left(\sum_{l=1}^{p} \alpha_l \widehat{Z}_{t-l} - \widehat{Z}_t\right)}{\widehat{\sigma}_t^4(\boldsymbol{\alpha})},
$$

and

$$\frac{\partial^2 \widehat{Q}_n(\boldsymbol{\alpha})}{\partial \alpha_k \partial \alpha_l} = -(n-p)^{-1} \sum_{t=p+1}^{n} \frac{\widehat{Z}_{t-k}\widehat{Z}_{t-l}}{\widehat{\sigma}_t^4(\boldsymbol{\alpha})} + 2(n-p)^{-1} \sum_{t=p+1}^{n} \frac{\widehat{Z}_{t-k}\widehat{Z}_{t-l}\widehat{X}_t^2}{\widehat{\sigma}_t^6(\boldsymbol{\alpha})}.$$

Lemma A.5. *Under Assumptions (a), (b) and (d), for* $\mathbf{M}$ *defined in (2.5), as* $n \to \infty$

$$\begin{aligned} & \left.\frac{\partial \widehat{Q}_n(\boldsymbol{\alpha})}{\partial \alpha_k}\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0} - \left.\frac{\partial Q_n(\boldsymbol{\alpha})}{\partial \alpha_k}\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0} \\ = & -\omega_0 \mathsf{E}\left\{\frac{Z_{p+1-k}}{\sigma_{p+1}^4(\boldsymbol{\alpha}_0)}\right\}(n-p)^{-1}\sum_{t=p+1}^{n} Z_t + o_p\left(n^{-1/2}\right) \\ = & \ \mathcal{O}_p\left(n^{-1/2}\right). \end{aligned} \tag{A.21}$$

Proof. Applying Lemmas A.9 and A.11, (A.17) and (A.19), one obtains that for any $1 \le k \le p$,

$$\begin{aligned} & \left.\frac{\partial \widehat{Q}_n(\boldsymbol{\alpha})}{\partial \alpha_k}\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0} - \left.\frac{\partial Q_n(\boldsymbol{\alpha})}{\partial \alpha_k}\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0} \\ = & \ (n-p)^{-1}\sum_{t=p+1}^{n}\left\{\frac{\widehat{Z}_{t-k}\left(\sum_{l=1}^{p}\alpha_{0,l}\widehat{Z}_{t-l}-\widehat{Z}_t\right)}{\widehat{\sigma}_t^4(\boldsymbol{\alpha}_0)} - \frac{Z_{t-k}\left(\sum_{l=1}^{p}\alpha_{0,l}Z_{t-l}-Z_t\right)}{\sigma_t^4(\boldsymbol{\alpha}_0)}\right\} \\ = & \ (n-p)^{-1}\sum_{t=p+1}^{n}\frac{\widehat{Z}_{t-k}\left(\sum_{l=1}^{p}\alpha_{0,l}\widehat{Z}_{t-l}-\widehat{Z}_t\right) - Z_{t-k}\left(\sum_{l=1}^{p}\alpha_{0,l}Z_{t-l}-Z_t\right)}{\sigma_t^4(\boldsymbol{\alpha}_0)} \\ & -(n-p)^{-1}\sum_{t=p+1}^{n}\frac{Z_{t-k}\left(\sum_{l=1}^{p}\alpha_{0,l}Z_{t-l}-Z_t\right)\left\{\widehat{\sigma}_t^4(\boldsymbol{\alpha}_0)-\sigma_t^4(\boldsymbol{\alpha}_0)\right\}}{\sigma_t^8(\boldsymbol{\alpha}_0)} + \mathcal{O}_p\left(\Delta_{n,0}^2\right), \end{aligned}$$

where one has taken advantage of the (A.15) and (A.16). Next,

$$\begin{aligned} & \left.\frac{\partial \widehat{Q}_n(\boldsymbol{\alpha})}{\partial \alpha_k}\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0} - \left.\frac{\partial Q_n(\boldsymbol{\alpha})}{\partial \alpha_k}\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0} \\ = & \ (n-p)^{-1}\sum_{t=p+1}^{n}\frac{Z_{t-k}\left\{\sum_{l=1}^{p}\alpha_{0,l}\left(\widehat{Z}_{t-l}-Z_{t-l}\right)-\widehat{Z}_t+Z_t\right\}}{\sigma_t^4(\boldsymbol{\alpha}_0)} \\ & +(n-p)^{-1}\sum_{t=p+1}^{n}\frac{\left(\widehat{Z}_{t-k}-Z_{t-k}\right)\left(\sum_{l=1}^{p}\alpha_{0,l}Z_{t-l}-Z_t\right)}{\sigma_t^4(\boldsymbol{\alpha}_0)} \\ & -(n-p)^{-1}\sum_{t=p+1}^{n}\frac{2Z_{t-k}\left(\sum_{l=1}^{p}\alpha_{0,l}Z_{t-l}-Z_t\right)\sum_{l=1}^{p}\alpha_{0,l}\left(\widehat{Z}_{t-l}-Z_{t-l}\right)}{\sigma_t^6(\boldsymbol{\alpha}_0)} \\ & +\mathcal{O}_p\left(\Delta_{n,0}^2\right). \end{aligned}$$

Thus

$$
\begin{aligned}
&\left.\frac{\partial \widehat{Q}_n(\boldsymbol{\alpha})}{\partial \alpha_k}\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0} - \left.\frac{\partial Q_n(\boldsymbol{\alpha})}{\partial \alpha_k}\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0} \\
= \;& (n-p)^{-1} \sum_{t=p+1}^{n} \frac{Z_{t-k}\left\{\sum_{l=1}^{p} \alpha_{0,l}\left(\widehat{Z}_{t-l} - Z_{t-l}\right) - \widehat{Z}_t + Z_t\right\}}{\sigma_t^4(\boldsymbol{\alpha}_0)} \\
& - (n-p)^{-1} \sum_{t=p+1}^{n} \frac{\left(\widehat{Z}_{t-k} - Z_{t-k}\right)\xi_t}{\sigma_t^4(\boldsymbol{\alpha}_0)} \\
& + 2\,(n-p)^{-1} \sum_{t=p+1}^{n} \frac{Z_{t-k}\xi_t \sum_{l=1}^{p} \alpha_{0,l}\left(\widehat{Z}_{t-l} - Z_{t-l}\right)}{\sigma_t^6(\boldsymbol{\alpha}_0)} + \mathcal{O}_p\left(\Delta_{n,0}^2\right).
\end{aligned}
$$

Applying (A.1) and (A.4) to the above,

$$
\begin{aligned}
&\left.\frac{\partial \widehat{Q}_n(\boldsymbol{\alpha})}{\partial \alpha_k}\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0} - \left.\frac{\partial Q_n(\boldsymbol{\alpha})}{\partial \alpha_k}\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0} \\
= \;& -(n-p)^{-1} \sum_{t=p+1}^{n} \frac{Z_{t-k}\left(Z_t+1\right)}{\sigma_t^4(\boldsymbol{\alpha}_0)} \frac{g\left(\frac{t}{n}\right) - \hat{g}\left(\frac{t}{n}\right)}{g\left(\frac{t}{n}\right)} \\
& + \sum_{l=1}^{p} \alpha_{0,l}\,(n-p)^{-1} \sum_{t=p+1}^{n} \frac{Z_{t-k}\left(Z_{t-l}+1\right)}{\sigma_t^4(\boldsymbol{\alpha}_0)} \frac{g\left(\frac{t-l}{n}\right) - \hat{g}\left(\frac{t-l}{n}\right)}{g\left(\frac{t-l}{n}\right)} \\
& - (n-p)^{-1} \sum_{t=p+1}^{n} \frac{\xi_t\left(Z_{t-k}+1\right)}{\sigma_t^4(\boldsymbol{\alpha}_0)} \frac{g\left(\frac{t-k}{n}\right) - \hat{g}\left(\frac{t-k}{n}\right)}{g\left(\frac{t-k}{n}\right)} \\
& + 2\sum_{l=1}^{p} \alpha_{0,l}\,(n-p)^{-1} \sum_{t=p+1}^{n} \frac{Z_{t-k}\xi_t\left(Z_{t-l}+1\right)}{\sigma_t^6(\boldsymbol{\alpha}_0)} \frac{g\left(\frac{t-l}{n}\right) - \hat{g}\left(\frac{t-l}{n}\right)}{g\left(\frac{t-l}{n}\right)} + \mathcal{O}_p\left(\Delta_{n,0}^2\right).
\end{aligned}
$$

Now applying (A.3) and (A.6), one obtains that

$$
\begin{aligned}
&\left.\frac{\partial \widehat{Q}_n(\boldsymbol{\alpha})}{\partial \alpha_k}\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0} - \left.\frac{\partial Q_n(\boldsymbol{\alpha})}{\partial \alpha_k}\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0} \\
= & -(n-p)^{-1} \sum_{t=p+1}^{n} \frac{Z_{t-k}\left(Z_t+1\right)}{\sigma_t^4(\boldsymbol{\alpha}_0)} \frac{g\left(\frac{t}{n}\right)-\hat{g}\left(\frac{t}{n}\right)}{g\left(\frac{t}{n}\right)} \\
& + \sum_{l=1}^{p} \alpha_{0,l}\,(n-p)^{-1} \sum_{t=p+1}^{n} \frac{Z_{t-k}\left(Z_{t-l}+1\right)}{\sigma_t^4(\boldsymbol{\alpha}_0)} \frac{g\left(\frac{t}{n}\right)-\hat{g}\left(\frac{t}{n}\right)}{g\left(\frac{t}{n}\right)} \\
& -(n-p)^{-1} \sum_{t=p+1}^{n} \frac{\xi_t\left(Z_{t-k}+1\right)}{\sigma_t^4(\boldsymbol{\alpha}_0)} \frac{g\left(\frac{t}{n}\right)-\hat{g}\left(\frac{t}{n}\right)}{g\left(\frac{t}{n}\right)} \\
& +2 \sum_{l=1}^{p} \alpha_{0,l}\,(n-p)^{-1} \sum_{t=p+1}^{n} \frac{Z_{t-k}\xi_t\left(Z_{t-l}+1\right)}{\sigma_t^6(\boldsymbol{\alpha}_0)} \frac{g\left(\frac{t}{n}\right)-\hat{g}\left(\frac{t}{n}\right)}{g\left(\frac{t}{n}\right)} \\
& +\mathcal{O}_p\left(\Delta_{n,0}^2+n^{-r}\right).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\left.\frac{\partial \widehat{Q}_n(\boldsymbol{\alpha})}{\partial \alpha_k}\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0} - \left.\frac{\partial Q_n(\boldsymbol{\alpha})}{\partial \alpha_k}\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0} \\
= & (n-p)^{-1} \sum_{t=p+1}^{n} \frac{Z_{t-k}\left(\sum_{l=1}^{p} \alpha_{0,l} Z_{t-l} - Z_t + \sum_{l=1}^{p} \alpha_{0,l} - 1\right)}{\sigma_t^4(\boldsymbol{\alpha}_0)} \frac{g\left(\frac{t}{n}\right)-\hat{g}\left(\frac{t}{n}\right)}{g\left(\frac{t}{n}\right)} \\
& -(n-p)^{-1} \sum_{t=p+1}^{n} \frac{\xi_t\left(Z_{t-k}+1\right)}{\sigma_t^4(\boldsymbol{\alpha}_0)} \frac{g\left(\frac{t}{n}\right)-\hat{g}\left(\frac{t}{n}\right)}{g\left(\frac{t}{n}\right)} \\
& +2 \sum_{l=1}^{p} \alpha_{0,l}\,(n-p)^{-1} \sum_{t=p+1}^{n} \frac{Z_{t-k}\xi_t\left(Z_{t-l}+1\right)}{\sigma_t^6(\boldsymbol{\alpha}_0)} \frac{g\left(\frac{t}{n}\right)-\hat{g}\left(\frac{t}{n}\right)}{g\left(\frac{t}{n}\right)} \\
& +\mathcal{O}_p\left(\Delta_{n,0}^2+n^{-r}\right) \\
= & \left(\sum_{l=1}^{p} \alpha_{0,l} - 1\right) \mathsf{E}\left\{\frac{Z_{p+1-k}}{\sigma_{p+1}^4(\boldsymbol{\alpha}_0)}\right\} (n-p)^{-1} \sum_{t=p+1}^{n} \frac{g\left(\frac{t}{n}\right)-\hat{g}\left(\frac{t}{n}\right)}{g\left(\frac{t}{n}\right)} \\
& + \left(\sum_{l=1}^{p} \alpha_{0,l} - 1\right) (n-p)^{-1} \sum_{t=p+1}^{n} \left[\frac{Z_{t-k}}{\sigma_t^4(\boldsymbol{\alpha}_0)} - \mathsf{E}\left\{\frac{Z_{p+1-k}}{\sigma_{p+1}^4(\boldsymbol{\alpha}_0)}\right\}\right] \frac{g\left(\frac{t}{n}\right)-\hat{g}\left(\frac{t}{n}\right)}{g\left(\frac{t}{n}\right)} \\
& +o_p\left(n^{-1/2}\right),
\end{aligned}
$$

as $\mathcal{O}_p\left(\Delta_{n,0}^2+n^{-r}\right) = o_p\left(n^{-1/2}\right)$ according to (2.10) and (2.12) in Assumption (c).

Note next that

$$
\begin{aligned}
&= \left(\sum_{l=1}^{p}\alpha_{0,l}-1\right)\mathsf{E}\left\{\frac{Z_{p+1-k}}{\sigma_{p+1}^4(\boldsymbol{\alpha}_0)}\right\}(n-p)^{-1}\sum_{t=p+1}^{n}\frac{g\left(\frac{t}{n}\right)-\hat{g}\left(\frac{t}{n}\right)}{g\left(\frac{t}{n}\right)} \\
&\quad+\left(\sum_{l=1}^{p}\alpha_{0,l}-1\right)(n-p)^{-1}\sum_{t=p+1}^{n}\left[\frac{Z_{t-k}}{\sigma_t^4(\boldsymbol{\alpha}_0)}-\mathsf{E}\left\{\frac{Z_{p+1-k}}{\sigma_{p+1}^4(\boldsymbol{\alpha}_0)}\right\}\right]\frac{g\left(\frac{t}{n}\right)-\hat{g}\left(\frac{t}{n}\right)}{g\left(\frac{t}{n}\right)} \\
&\quad+o_p\left(n^{-1/2}\right) \\
&= -\omega_0\left[\mathsf{E}\left\{\frac{Z_{p+1-k}}{\sigma_{p+1}^4(\boldsymbol{\alpha}_0)}\right\}\right](n-p)^{-1}\sum_{t=p+1}^{n}Z_t+o_p\left(n^{-1/2}\right).
\end{aligned}
$$

The proof is complete by noting that $(n-p)^{-1}\sum_{t=p+1}^{n}Z_t=\mathcal{O}_p\left(n^{-1/2}\right)$ because $\{Z_t\}_{t=-\infty}^{+\infty}$ is strictly stationary with mean zero and finite variance according to Assumption (b). □

Lemma A.6. *Under Assumptions (a)-(c), as $n\to\infty$,*

$$\sup_{\boldsymbol{\alpha}\in\boldsymbol{\Xi}}\left|\frac{\partial^2\widehat{Q}_n(\boldsymbol{\alpha})}{\partial\boldsymbol{\alpha}\partial\boldsymbol{\alpha}^T}-\frac{\partial^2 Q_n(\boldsymbol{\alpha})}{\partial\boldsymbol{\alpha}\partial\boldsymbol{\alpha}^T}\right|=\mathcal{O}_p(\Delta_{n,0})=o_p\left(n^{-1/4}\right).$$

Proof. The proof is similar to that of Corollary A.1. □

Lemma A.7. *Under Assumptions (a) and (b), as $n\to\infty$,*

$$\left.\frac{\partial^2 Q_n(\boldsymbol{\alpha})}{\partial\boldsymbol{\alpha}\partial\boldsymbol{\alpha}^T}\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0}-\mathbf{J}=\mathcal{O}_p\left(n^{-1/2}\right).$$

Proof. Note by (A.20) and (2.15)

$$
\begin{aligned}
&\left.\frac{\partial^2 Q_n(\boldsymbol{\alpha})}{\partial\alpha_k\partial\alpha_l}\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0} \\
&= (n-p)^{-1}\sum_{t=p+1}^{n}\frac{Z_{t-k}Z_{t-l}\left\{2X_t^2-\sigma_t^2(\boldsymbol{\alpha}_0)\right\}}{\sigma_t^6(\boldsymbol{\alpha}_0)} \\
&= 2(n-p)^{-1}\sum_{t=p+1}^{n}\frac{Z_{t-k}Z_{t-l}\xi_t}{\sigma_t^6(\boldsymbol{\alpha}_0)}+(n-p)^{-1}\sum_{t=p+1}^{n}\frac{Z_{t-k}Z_{t-l}}{\sigma_t^4(\boldsymbol{\alpha}_0)} \\
&= \mathsf{E}\frac{Z_{t-k}Z_{t-l}}{\sigma_t^4(\boldsymbol{\alpha}_0)}+2(n-p)^{-1}\sum_{t=p+1}^{n}\frac{Z_{t-k}Z_{t-l}\xi_t}{\sigma_t^6(\boldsymbol{\alpha}_0)} \\
&\quad+(n-p)^{-1}\sum_{t=p+1}^{n}\left(\frac{Z_{t-k}Z_{t-l}}{\sigma_t^4(\boldsymbol{\alpha}_0)}-\mathsf{E}\frac{Z_{t-k}Z_{t-l}}{\sigma_t^4(\boldsymbol{\alpha}_0)}\right),
\end{aligned}
$$

thus

$$\left.\frac{\partial^2 Q_n(\boldsymbol{\alpha})}{\partial\boldsymbol{\alpha}\partial\boldsymbol{\alpha}^{\mathrm{T}}}\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0} - \mathbf{J} = \mathcal{O}_p\left(n^{-1/2}\right).$$

A.3 Proof of Theorem 3.1

In this subsection, we verify that the unintuitive set of complicated inequalities in Assumptions (c) and (d) are satisfied by B-spline estimators under the elementary Assumptions (c1) and (c2). Denote by $\|\mathbf{A}\|_\infty = \max(|a_{i,j}|)$ for any matrix $\mathbf{A} = (a_{i,j})$, and $\lambda_{\max}(\mathbf{A})$ the largest eigenvalue of any real symmetric matrix $\mathbf{A}$. According to Equation (S.3) in Lemma S.1 of Shao and Yang (2017), there exist constants $C_\lambda, C > 0$ such that

$$\left\|\left((n-p)^{-1}\mathbf{B}_m^{\mathrm{T}}\mathbf{B}_m\right)^{-1}\right\|_\infty \le C_\lambda, \lambda_{\max}\left((n-p)^{-1}\mathbf{B}_m^{\mathrm{T}}\mathbf{B}_m\right)^{-1} \le C_\lambda, \|\mathbf{B}_m\|_\infty \le CN^{1/2}. \tag{A.22}$$

According to (S.2) in Shao and Yang (2017)

$$\max_{p+1\le t\le n}\left|\tilde{g}_m(t/n) - g(t/n)\right| \le \sup_{u\in[0,1]}\left|\tilde{g}_m(u) - g(u)\right| = \mathcal{O}\left(N^{-m'-\delta}\right) \tag{A.23}$$

in which

$$\begin{aligned}\tilde{g}_m(u) &= \mathbf{B}_m(u)\left(\mathbf{B}_m^{\mathrm{T}}\mathbf{B}_m\right)^{-1}\mathbf{B}_m^{\mathrm{T}}\mathbf{g}, u\in[0,1]\\ \mathbf{B}_m &= \left\{\mathbf{B}_m^{\mathrm{T}}((p+1)/n), \ldots, \mathbf{B}_m^{\mathrm{T}}(n/n)\right\}^{\mathrm{T}},\\ \mathbf{g} &= \left\{g((p+1)/n), \ldots, g(n/n)\right\}^{\mathrm{T}}.\end{aligned}$$

In addition, the standard B spline theory implies

$$\begin{aligned}\sup_{u\in[0,1]}\left|\tilde{g}_m(u) - \hat{g}_m(u)\right| &= \sup_{u\in[0,1]}\left|\mathbf{B}_m(u)\left(\mathbf{B}_m^{\mathrm{T}}\mathbf{B}_m\right)^{-1}\mathbf{B}_m^{\mathrm{T}}\left(\mathbf{Y}_n^2 - \mathbf{g}\right)\right|\\ &= \mathcal{O}_p\left(n^{-1/2}N^{1/2}\log n\right),\end{aligned} \tag{A.24}$$

which is proved by showing

$$\left\|(n-p)^{-1}\mathbf{B}_m^{\mathrm{T}}\left(\mathbf{Y}_n^2 - \mathbf{g}\right)\right\|_\infty = \mathcal{O}_p\left(n^{-1/2}\log n\right)$$

with the same exponential inequality as in Lemma A.7 of Wang and Yang (2009). We denote for any functions ϕ, ψ defined on $[0,1]$ the following empirical inner product and empirical norm

$$(\phi,\psi)_n = (n-p)^{-1}\sum_{t=p+1}^{n}\phi(t/n)\psi(t/n), \|\phi\|_2 = \left((\phi,\phi)_n\right)^{1/2},$$

and note the following fact: for any bounded function w

$$\max_{-m+1\leq j\leq N}\left|\left(B_{j,m},w\right)_n\right|\leq CN^{-1/2}. \tag{A.25}$$

Lemma A.8. *Under Assumptions (a), (b), (c1) and (c2), as $n\to\infty$*

$$\Delta_{n,0}\left(\hat{g}_m\right)=\mathcal{O}_p\left(n^{-1/2}N^{1/2}\log n+N^{-m'-\delta}\right)=o_p\left(n^{-1/4}\right), \tag{A.26}$$

$$\Delta_{n,1}\left(\hat{g}_m\right)=\mathcal{O}_p\left(n^{-1}N\log^2 n+N^{-2m'-2\delta}\right)=o_p\left(n^{-1/2}\right), \tag{A.27}$$

$$\Delta_{n,2}\left(\hat{g}_m\right)=\mathcal{O}_p\left(n^{-1/2}N^{1/2}\log n+N^{-m'-\delta}\right)=o_p\left(n^{-1/4}\right). \tag{A.28}$$

Proof. Note that

$$\begin{aligned}\Delta_{n,0}\left(\hat{g}_m\right) &= \max_{p+1\leq t\leq n}\left|\hat{g}_m\left(t/n\right)-g\left(t/n\right)\right| \\ &\leq \sup_{u\in[0,1]}\left|\tilde{g}_m\left(u\right)-g\left(u\right)\right|+\sup_{u\in[0,1]}\left|\tilde{g}_m\left(u\right)-\hat{g}_m\left(u\right)\right|.\end{aligned}$$

Hence (A.26) follows from (A.23) and (A.24). Moreover, (A.27) and (A.28) follow from (A.26), (A.4) and (A.5). □

Lemma A.9. *Under Assumption (c1) and $\{\zeta_t\}_{t=p+1}^n$ is $\left(C_\zeta,\rho_\zeta\right)$-exponential correlated,*

$$(n-p)^{-1}\sum_{t=p+1}^{n}\zeta_t\frac{\tilde{g}_m\left(t/n\right)-g\left(t/n\right)}{g\left(t/n\right)}=\mathcal{O}_p\left(n^{-1/2}N^{-m'-\delta}\right)=o_p\left(n^{-1/2}\right). \tag{A.29}$$

Proof. According to the definition (2.1), $\{\zeta_t\}_{t=p+1}^n$ is (C,ρ)-exponential correlated for some constants $\left(C_\zeta,\rho_\zeta\right)$. Lemma 2 in Shao and Yang (2017) entails that

$$\begin{aligned}&\mathsf{E}\left[(n-p)^{-1}\sum_{t=p+1}^{n}\zeta_t\frac{\tilde{g}_m\left(t/n\right)-g\left(t/n\right)}{g\left(t/n\right)}\right]^2 \\ \leq\ & C_\zeta\left(1-\rho_\zeta\right)^{-1}(n-p)^{-1}\max_{p+1\leq t\leq n}\left\{\frac{\tilde{g}_m\left(t/n\right)-g\left(t/n\right)}{g\left(t/n\right)}\right\}^2 \\ \leq\ & C_\zeta\left(1-\rho_\zeta\right)^{-1}(n-p)^{-1}c_l^{-2}\times\mathcal{O}\left(N^{-2m'-2\delta}\right),\end{aligned}$$

where the last step follows from (A.23). Thus (A.29) follows by applying the Markov inequality. Proof of Lemma A.9 is thus complete. □

In what follows, we denote by $\mathrm{TV}\left[0,1\right]$ the space of functions of finite total variations on $[0,1]$, and for any $w\in\mathrm{TV}\left[0,1\right]$, $\mathrm{TV}_0^1\left(w\right)$ its total variation on $[0,1]$.

Lemma A.10. *Under Assumptions (c1) and (c2), there exists* $\phi \in G_N^{(m-2)}[0,1]$ *such that for some universal constant* $C_{m'+1+\delta} > 0$

$$\|g-\phi\|_{\infty} \leq C_{m'+1+\delta} \|g\|_{m',\delta} N^{-m'-\delta}, \|g-\phi\|_{0,r} \leq C_{m'+1+\delta}, \tag{A.30}$$

and the following hold:

(a) For any $w \in \mathrm{TV}[0,1]$

$$\left|\int_0^1 \{g(u)-\phi(u)\} w(u)\, du\right| \leq C N^{-m'-\delta-1} \mathrm{TV}_0^1(w). \tag{A.31}$$

In particular

$$\left|\int_0^1 \{g(u)-\phi(u)\} g^{-1}(u)\, du\right| \leq C N^{-m'-\delta-1}, \tag{A.32}$$

and for standardized B spline basis $B_{j,m}$

$$\max_{-m+1\leq j\leq N}\left|\int_0^1 \{g(u)-\phi(u)\} B_{j,m}(u)\, du\right| \leq C N^{1/2} N^{-m'-\delta-1}. \tag{A.33}$$

(b) There is absolute constant $C > 0$ *such that*

$$\left|(n-p)^{-1}\sum_{t=p+1}^{n}\{g(t/n)-\phi(t/n)\} w(t/n) - \int_0^1 \{g(u)-\phi(u)\} w(u)\, du\right|$$

$$\leq C\left[\|w\|_{\infty}\left\{n^{-1}N^{-m'-\delta} + |\{\mathrm{supp}(w)\}| \times n^{-r}\right\} + N^{-m'-\delta}\sum_{t=p+1}^{n}\int_{(t-1)/n}^{t/n}|w(t/n)-w(u)|\, du\right]. \tag{A.34}$$

In particular

$$\left|(n-p)^{-1}\sum_{t=p+1}^{n}\frac{g(t/n)-\phi(t/n)}{g(t/n)} - \int_0^1 \frac{g(u)-\phi(u)}{g(u)} du\right| \leq C\left(n^{-1}N^{-m'-\delta} + n^{-r}\right), \tag{A.35}$$

$$\max_{-m+1\leq j\leq N}\left|(n-p)^{-1}\sum_{t=p+1}^{n}\{g(t/n)-\phi(t/n)\} B_{j,m}(t/n) - \int_0^1 \{g(u)-\phi(u)\} B_{j,m}(u)\, du\right|$$

$$\leq C\left(Nn^{-1}N^{-m'-\delta-1/2} + N^{-1/2}n^{-r}\right). \tag{A.36}$$

(c) There exists a universal constant $C > 0$ *such that*

$$\left|(n-p)^{-1}\sum_{t=p+1}^{n}\frac{g(t/n)-\phi(t/n)}{g(t/n)}\right| \leq C\left(N^{-m'-\delta-1} + n^{-r}\right) \tag{A.37}$$

$$\max_{-m+1\leq j\leq N}\left|(n-p)^{-1}\sum_{t=p+1}^{n}\left\{g\left(t/n\right)-\phi\left(t/n\right)\right\}B_{j,m}\left(t/n\right)\right|\leq C\left(N^{-m'-\delta-1/2}+N^{-1/2}n^{-r}\right). \tag{A.38}$$

(d) There exists a universal constant $C>0$ such that

$$\left|(n-p)^{-1}\sum_{t=p+1}^{n}\frac{g\left(t/n\right)-\tilde{g}_m\left(t/n\right)}{g\left(t/n\right)}\right|\leq C\left(N^{-m'-\delta-1/2}+n^{-r}\right). \tag{A.39}$$

Proof. Define the integral function $G\left(x\right)=\int_0^x g\left(u\right)du$, then $G'=g, G\in C^{(m'+1,\delta)}\left[0,1\right]$, and according to Theorem 7.7.4 in DeVore and Lorentz (1993), for some universal constant $C_{m'+1+\delta}>0$, and some quasi-interpolant $\Phi\in G_N^{(m-1)}[0,1]$ of G

$$\begin{aligned}\left\|G-\Phi\right\|_\infty &\leq C_{m'+1+\delta}\left\|G\right\|_{m'+1,\delta}N^{-m'-1-\delta}, \Phi\left(0\right)=G\left(0\right), \Phi\left(1\right)=G\left(1\right)\\ \left\|g-\phi\right\|_\infty &\leq C_{m'+1+\delta}\left\|g\right\|_{m',\delta}N^{-m'-\delta}, \left\|g-\phi\right\|_{0,r}\leq C_{m'+1+\delta},\end{aligned}$$

in which $\phi=\Phi'\in G_N^{(m-2)}[0,1]$. Consequently

$$\left|\int_0^1\left\{g\left(u\right)-\phi\left(u\right)\right\}w\left(u\right)du\right|=$$

$$\left|\left\{G\left(u\right)-\Phi\left(u\right)\right\}w\left(u\right)|_0^1-\int_0^1\left\{G\left(u\right)-\Phi\left(u\right)\right\}dw\left(u\right)\right|$$

$$\leq\left\|G-\Phi\right\|_\infty \mathrm{TV}_0^1\left(w\right)\leq C_{m'+1+\delta}\left\|G\right\|_{m'+1,\delta}N^{-m'-\delta-1}\mathrm{TV}_0^1\left(w\right),$$

which proves (A.31). Next, (A.33) and (A.32) follow by noticing $\max_{-m+1\leq j\leq N}\mathrm{TV}_0^1\left(B_{j,m}\right)=\mathcal{O}\left(N^{1/2}\right)$ and $\mathrm{TV}_0^1\left(g^{-1}\right)<\infty$, so (a) is proved.

To prove (b), note that

$$\left|(n-p)^{-1}\sum_{t=p+1}^{n}\left\{g\left(t/n\right)-\phi\left(t/n\right)\right\}w\left(t/n\right)-\int_0^1\left\{g\left(u\right)-\phi\left(u\right)\right\}w\left(u\right)du\right|$$

$$\begin{aligned}\leq &\int_0^{p/n}\left|g\left(u\right)-\phi\left(u\right)\right|\left|w\left(u\right)\right|du+\left|(n-p)^{-1}-n^{-1}\right|\sum_{t=p+1}^{n}w\left(t/n\right)\left|g\left(t/n\right)-\phi\left(t/n\right)\right|\\ &+\left|\sum_{t=p+1}^{n}\int_{(t-1)/n}^{t/n}\left[\left\{g\left(t/n\right)-\phi\left(t/n\right)\right\}w\left(t/n\right)-\left\{g\left(u\right)-\phi\left(u\right)\right\}w\left(u\right)\right]du\right|\end{aligned}$$

$$\leq \; Cn^{-1}N^{-m'-\delta}\left\|w\right\|_{\infty} + \left|\sum_{t=p+1}^{n}\int_{(t-1)/n}^{t/n}\left\{g\left(t/n\right)-\phi\left(t/n\right)\right\}\left\{w\left(t/n\right)-w\left(u\right)\right\}du\right|$$
$$+\left|\sum_{t=p+1}^{n}\int_{(t-1)/n}^{t/n}\left[\left\{g\left(t/n\right)-\phi\left(t/n\right)\right\}-\left\{g\left(u\right)-\phi\left(u\right)\right\}\right]w\left(u\right)du\right|$$

$$\leq C\left[\left\|w\right\|_{\infty}\left\{n^{-1}N^{-m'-\delta}+\left|\left\{\mathrm{supp}\left(w\right)\right\}\right|\times n^{-r}\right\}+N^{-m'-\delta}\sum_{t=p+1}^{n}\int_{(t-1)/n}^{t/n}\left|w\left(t/n\right)-w\left(u\right)\right|du\right],$$

which proves (A.34). In particular, if $w = g^{-1}$, then $\left\|w\right\|_{\infty}, \left|\left\{\mathrm{supp}\left(w\right)\right\}\right|$ are all bounded and $\int_{(t-1)/n}^{t/n}\left|g^{-1}\left(t/n\right)-g^{-1}\left(u\right)\right|du \leq n^{-1}\left\|g^{-1}\right\|_{0,r} n^{-r}$, thus

$$\left|\left(n-p\right)^{-1}\sum_{t=p+1}^{n}\frac{g\left(t/n\right)-\phi\left(t/n\right)}{g\left(t/n\right)}-\int_{0}^{1}\frac{g\left(u\right)-\phi\left(u\right)}{g\left(u\right)}du\right|$$

$$\leq C\left[n^{-1}N^{-m'-\delta}+n^{-r}+N^{-m'-\delta}\sum_{t=p+1}^{n}n^{-1}\left\|g^{-1}\right\|_{0,r}n^{-r}\right],$$
$$\leq C\left(n^{-1}N^{-m'-\delta}+n^{-r}+N^{-m'-\delta}n^{-r}\right)\leq C\left(n^{-1}N^{-m'-\delta}+n^{-r}\right),$$

proving (A.35). If $w = B_{j,m}$, then by definition $\max_{1-m\leq j\leq N}\left|\left\{\mathrm{supp}\left(B_{j,m}\right)\right\}\right| \leq CN^{-1}$ and also $\max_{1-m\leq j\leq N}\left\|B_{j,m}\right\|_{\infty} \leq CN^{1/2}$. If $m > 1$, then w is Lipschitz continuous with $\max_{1-m\leq j\leq N}\left\|B_{j,m}\right\|_{0,1} \leq CNN^{1/2}$, and on all subintevals $\left[\left(t-1\right)/n, t/n\right]$ except at most CnN^{-1} of these, $B_{j,m} \equiv 0$, hence

$$\begin{aligned} N^{-m'-\delta}\sum_{t=p+1}^{n}\int_{(t-1)/n}^{t/n}\left|B_{j,m}\left(t/n\right)-B_{j,m}\left(u\right)\right|du &\leq CN^{-m'-\delta}nN^{-1}\times n^{-1}NN^{1/2}n^{-1} \\ &\leq CN^{-m'-\delta}n^{-1}N^{1/2}. \end{aligned}$$

If $w = B_{j,m}, m = 1$, then for all subintevals $\left[\left(t-1\right)/n, t/n\right]$ except at most 2 of these

$$B_{j,m}\left(t/n\right)-B_{j,m}\left(u\right)\equiv 0, u\in\left[\left(t-1\right)/n, t/n\right]$$

so

$$\begin{aligned} N^{-m'-\delta}\sum_{t=p+1}^{n}\int_{(t-1)/n}^{t/n}\left|B_{j,m}\left(t/n\right)-B_{j,m}\left(u\right)\right|du &\leq CN^{-m'-\delta}\times 2\times n^{-1}N^{1/2} \\ &\leq CN^{-m'-\delta}n^{-1}N^{1/2}. \end{aligned}$$

Putting all the above together

$$\max_{-m+1\le j\le N}\left|(n-p)^{-1}\sum_{t=p+1}^{n}\{g(t/n)-\phi(t/n)\}B_{j,m}(t/n)-\int_0^1\{g(u)-\phi(u)\}B_{j,m}(u)\,du\right|$$

$$\le C\left[N^{1/2}\left\{n^{-1}N^{-m'-\delta}+N^{-1}\times n^{-r}\right\}+N^{-m'-\delta}n^{-1}N^{1/2}\right]\le C\left(Nn^{-1}N^{-m'-\delta-1/2}+N^{-1/2}n^{-r}\right)$$

hence (A.36) is proved.

To prove (c), noting that (A.32) and (A.35) togther imply that

$$\left|(n-p)^{-1}\sum_{t=p+1}^{n}\frac{g(t/n)-\phi(t/n)}{g(t/n)}\right|\le C\left(N^{-m'-\delta-1}+n^{-1}N^{-m'-\delta}+n^{-r}\right)$$

and (A.37) follows as $n^{-1}\ll N^{-1}$. Likewise (A.38) follows from (A.33) and (A.36).

To prove (d), denote $\boldsymbol{\phi}=\{\phi((p+1)/n),,\phi(n/n)\}^{\mathrm{T}}$, then

$$g(u)-\tilde{g}_m(u)=g(u)-\phi(u)+\mathbf{B}_m(u)\left(\mathbf{B}_m^{\mathrm{T}}\mathbf{B}_m\right)^{-1}\mathbf{B}_m^{\mathrm{T}}(\boldsymbol{\phi}-\mathbf{g})$$

and that

$$\begin{aligned}\left|(n-p)^{-1}\sum_{t=p+1}^{n}\frac{g(t/n)-\tilde{g}_m(t/n)}{g(t/n)}\right| &\le \left|(n-p)^{-1}\sum_{t=p+1}^{n}\frac{g(t/n)-\phi(t/n)}{g(t/n)}\right|+\\ &\quad\left|(n-p)^{-1}\sum_{t=p+1}^{n}\frac{\mathbf{B}_m(t/n)}{g(t/n)}\left(\mathbf{B}_m^{\mathrm{T}}\mathbf{B}_m\right)^{-1}\mathbf{B}_m^{\mathrm{T}}(\boldsymbol{\phi}-\mathbf{g})\right|.\end{aligned}$$

Note next $\max_{-m+1\le j\le N}\left|\left\{\left\langle B_{j,m},g^{-1}\right\rangle_n\right\}\right|\le CN^{-1/2}$ according to (A.25) and that $\lambda_{\max}\left((n-p)^{-1}\mathbf{B}_m^{\mathrm{T}}\mathbf{B}_m\right)^{-1}\le C_\lambda$ by (A.22), hence

$$\left|(n-p)^{-1}\sum_{t=p+1}^{n}\frac{\mathbf{B}_m(t/n)}{g(t/n)}\left(\mathbf{B}_m^{\mathrm{T}}\mathbf{B}_m\right)^{-1}\mathbf{B}_m^{\mathrm{T}}(\boldsymbol{\phi}-\mathbf{g})\right|$$

$$=\left|\left\{\left\langle B_{j,m},g^{-1}\right\rangle_n\right\}_{-m+1\le j\le N}^{\mathrm{T}}\left(\mathbf{B}_m^{\mathrm{T}}\mathbf{B}_m/(n-p)\right)^{-1}\left\{\langle B_{j,m},\phi-g\rangle_n\right\}_{-m+1\le j\le N}^{\mathrm{T}}\right|$$

$$\begin{aligned}&\le\ C\sqrt{N\left(CN^{-1/2}\right)^2}\times C\times\sqrt{N\left(CN^{-1/2}\right)^2}\\ &\qquad\times\max_{-m+1\le j\le N}\left|(n-p)^{-1}\sum_{t=p+1}^{n}\{g(t/n)-\phi(t/n)\}B_{j,m}(t/n)\right|\end{aligned}$$

$$\le C\max_{-m+1\le j\le N}\left|(n-p)^{-1}\sum_{t=p+1}^{n}\{g(t/n)-\phi(t/n)\}B_{j,m}(t/n)\right|$$

$$\leq C\left(N^{-m'-\delta-1/2}+N^{-1/2}n^{-r}\right)$$

according to (A.38). Consequently

$$\left|(n-p)^{-1}\sum_{t=p+1}^{n}\frac{g\left(t/n\right)-\tilde{g}_m\left(t/n\right)}{g\left(t/n\right)}\right| \leq \left|(n-p)^{-1}\sum_{t=p+1}^{n}\frac{g\left(t/n\right)-\phi\left(t/n\right)}{g\left(t/n\right)}\right| +C\left(N^{-m'-\delta-1/2}+N^{-1/2}n^{-r}\right)$$

$$\leq C\left(N^{-m'-\delta-1}+n^{-r}\right)+C\left(N^{-m'-\delta-1/2}+N^{-1/2}n^{-r}\right)=C\left(N^{-m'-\delta-1/2}+n^{-r}\right)$$

which proves (A.39). □

Lemma A.11. *Under Assumptions (a), (b), (c1) and (c2),*

$$(n-p)^{-1}\sum_{t=p+1}^{n}\frac{g(t/n)-\hat{g}_m(t/n)}{g(t/n)} = (n-p)^{-1}\sum_{t=p+1}^{n}Z_t+\mathcal{O}_p\left(n^{-1/2}N^{-m'-\delta}\right)+\mathcal{O}\left(N^{-m'-\delta-1/2}+n^{-r}\right). \quad \text{(A.40)}$$

Therefore the B-spline estimator $\hat{g}_m$ satisfies Assumption (d).

Proof. According to (A.39),

$$\left|(n-p)^{-1}\sum_{t=p+1}^{n}\frac{g\left(t/n\right)-\tilde{g}_m\left(t/n\right)}{g\left(t/n\right)}\right|\leq C\left(N^{-m'-\delta-1/2}+n^{-r}\right)=\mathcal{O}\left(N^{-m'-\delta-1/2}+n^{-r}\right).$$

On the other hand,

$$\begin{aligned}
&(n-p)^{-1}\sum_{t=p+1}^{n}\frac{\hat{g}_m(t/n)-\tilde{g}_m(t/n)}{g(t/n)}\\
=\ &(n-p)^{-1}\left(\frac{1}{g\left((p+1)/n\right)},\ldots,\frac{1}{g\left(n/n\right)}\right)\mathbf{B}_m\left(\mathbf{B}_m^{\mathrm{T}}\mathbf{B}_m\right)^{-1}\mathbf{B}_m^{\mathrm{T}}\left(\mathbf{Y}_n^2-\mathbf{g}\right)\\
=\ &(n-p)^{-1}\left(\psi\left(\frac{p+1}{n}\right),\ldots,\psi\left(\frac{n}{n}\right)\right)\mathbf{B}_m\left(\mathbf{B}_m^{\mathrm{T}}\mathbf{B}_m\right)^{-1}\mathbf{B}_m^{\mathrm{T}}\left(\mathbf{Y}_n^2-\mathbf{g}\right),
\end{aligned}$$

in which one denotes $\psi\left(u\right)=g\left(u\right)^{-1}, u\in[0,1]$. Assumption (c1) implies that $\psi\left(\cdot\right)\in C^{m'+\delta}[0,1]$, and consequently

$$\sup_{u\in[0,1]}\left|\psi\left(u\right)-\tilde{\psi}_m\left(u\right)\right|=\mathcal{O}\left(N^{-m'-\delta}\right), \quad \text{(A.41)}$$

in which

$$\begin{aligned}\tilde{\psi}_m(u) &= \mathbf{B}_m(u)\left(\mathbf{B}_m^{\mathrm{T}}\mathbf{B}_m\right)^{-1}\mathbf{B}_m^{\mathrm{T}}\boldsymbol{\psi}, u\in[0,1]\\ \boldsymbol{\psi} &= \left(\psi\left(\frac{p+1}{n}\right),\ldots,\psi\left(\frac{n}{n}\right)\right)^{\mathrm{T}}.\end{aligned}$$

One therefore can write

$$\begin{aligned}&(n-p)^{-1}\sum_{t=p+1}^{n}\frac{\hat{g}_m(t/n)-\tilde{g}_m(t/n)}{g(t/n)}\\ =\ &(n-p)^{-1}\sum_{t=p+1}^{n}\tilde{\psi}_m(t/n)\left\{Y_{t,n}^2-g(t/n)\right\}\\ =\ &(n-p)^{-1}\sum_{t=p+1}^{n}\left\{\tilde{\psi}_m(t/n)-\psi(t/n)\right\}g(t/n)Z_t\\ &+(n-p)^{-1}\sum_{t=p+1}^{n}\psi(t/n)g(t/n)Z_t\\ =\ &(n-p)^{-1}\sum_{t=p+1}^{n}\left\{\tilde{\psi}_m(t/n)-\psi(t/n)\right\}g(t/n)Z_t+(n-p)^{-1}\sum_{t=p+1}^{n}Z_t.\end{aligned}$$

It is easily verified that $\{Z_t\}_{t=p+1}^{n}$ is an (C_Z,ρ_Z)-exponentially correlated sequence, thus Lemma 2 in Shao and Yang (2017) entails that

$$\begin{aligned}&\mathsf{E}\left[(n-p)^{-1}\sum_{t=p+1}^{n}\left\{\tilde{\psi}_m(t/n)-\psi(t/n)\right\}g(t/n)Z_t\right]^2\\ \le\ &C_Z(1-\rho_Z)^{-1}(n-p)^{-1}c_u^2\max_{p+1\le t\le n}\left\{\tilde{\psi}_m(t/n)-\psi(t/n)\right\}^2\\ &\le C_Z(1-\rho_Z)^{-1}(n-p)^{-1}c_u^2\times\mathcal{O}\left(N^{-2m'-2\delta}\right)\end{aligned}$$

where the last step follows from (A.41). Thus

$$(n-p)^{-1}\sum_{t=p+1}^{n}\left\{\tilde{\psi}_m(t/n)-\psi(t/n)\right\}g(t/n)Z_t=\mathcal{O}_p\left(n^{-1/2}N^{-m'-\delta}\right),$$

and the lemma is proved. □

Proof of Theorem 3.1. We verify Assumption (c) with $\hat{g}$ replaced by B-spline estimator $\hat{g}_m$ in $\Delta_{n,0}$, $S_{n,1}$, $S_{n,2}$, and $S_{n,3}$. First, (A.26) and (3.2) in Assumption (c2) together imply that $n^{-1/2}N^{1/2}\log n+N^{-m'-\delta}=o\left(n^{-1/4}\right)$, thus $\Delta_{n,0}\left(\hat{g}_m\right)=o_p\left(n^{-1/4}\right)$

and (2.10) is proved for $\hat{g}_m$. That the true trend function g is Hölder continuous of order $r > 1/2$ follows from Assumptions (c1), (c2). The Hölder continuity order of $\hat{g}_m$ follows from standard B-spline theory, and thus (2.12) is proved for $\hat{g}_m$.

We now prove (2.12) for any $\left(C_\zeta, \rho_\zeta\right)$-exponential correlated sequence $\{\zeta_t\}_{t=p+1}^n$. Applying (A.29),

$$\begin{aligned}
S_{n,\zeta} &= (n-p)^{-1} \sum_{t=p+1}^{n} \zeta_t \frac{\hat{g}_m(t/n) - \tilde{g}_m(t/n)}{g(t/n)} \\
&\quad + (n-p)^{-1} \sum_{t=p+1}^{n} \zeta_t \frac{\tilde{g}_m(t/n) - g(t/n)}{g(t/n)} \\
&= (n-p)^{-1} \sum_{t=p+1}^{n} \zeta_t \frac{\hat{g}_m(t/n) - \tilde{g}_m(t/n)}{g(t/n)} + \mathcal{O}_p\left(n^{-1/2} N^{-m'-\delta}\right) \\
&= (n-p)^{-2} \left(\frac{\zeta_{p+1}}{g((p+1)/n)}, \ldots, \frac{\zeta_n}{g(n/n)} \right) \\
&\quad \times \mathbf{B}_m \left(\frac{\mathbf{B}_m^{\mathrm{T}} \mathbf{B}_m}{n-p} \right)^{-1} \mathbf{B}_m^{\mathrm{T}} \left(\mathbf{Y}_n^2 - \mathbf{g} \right) \\
&\quad + \mathcal{O}_p\left(n^{-1/2} N^{-m'-\delta}\right).
\end{aligned}$$

Lemma S.2 in Shao and Yang (2017) implies that for some constant $C > 0$,

$$\mathsf{E} \left\| \left(\frac{\zeta_{p+1}}{g((p+1)/n)}, \ldots, \frac{\zeta_n}{g(n/n)} \right) \mathbf{B}_m \right\|^2 \le CnN.$$

Hence

$$(n-p)^{-1} \left\| \left(\frac{\zeta_{p+1}}{g((p+1)/n)}, \ldots, \frac{\zeta_n}{g(n/n)} \right) \mathbf{B}_m \right\| = \mathcal{O}_p\left((n-p)^{-1} n^{1/2} N^{1/2} \right).$$

Then the Cauchy-Schwartz inequality entails that

$$\begin{aligned}
&\left| (n-p)^{-2} \left(\frac{\zeta_{p+1}}{g((p+1)/n)}, \ldots, \frac{\zeta_n}{g(n/n)} \right) \mathbf{B}_m \left(\frac{\mathbf{B}_m^{\mathrm{T}} \mathbf{B}_m}{n-p} \right)^{-1} \mathbf{B}_m^{\mathrm{T}} \left(\mathbf{Y}_n^2 - \mathbf{g} \right) \right| \\
&\quad \le \mathcal{O}_p\left((n-p)^{-1} n^{1/2} N^{1/2} \right) \times C_\lambda \times \mathcal{O}_p\left(n^{-1/2} \log n \right) \times (N+m)^{1/2} \\
&\quad = \mathcal{O}_p\left(n^{-1} N \log n \right) = o_p\left(n^{-1/2} \right).
\end{aligned}$$

Summing up the above, (2.12) is proved as

$$|S_{n,\zeta}| \le \mathcal{O}_p\left(n^{-1} N \log n \right) + \mathcal{O}_p\left(n^{-1/2} N^{-m'-\delta} \right) = o_p\left(n^{-1/2} \right).$$

The proof of Theorem 3.1 is thus complete. □

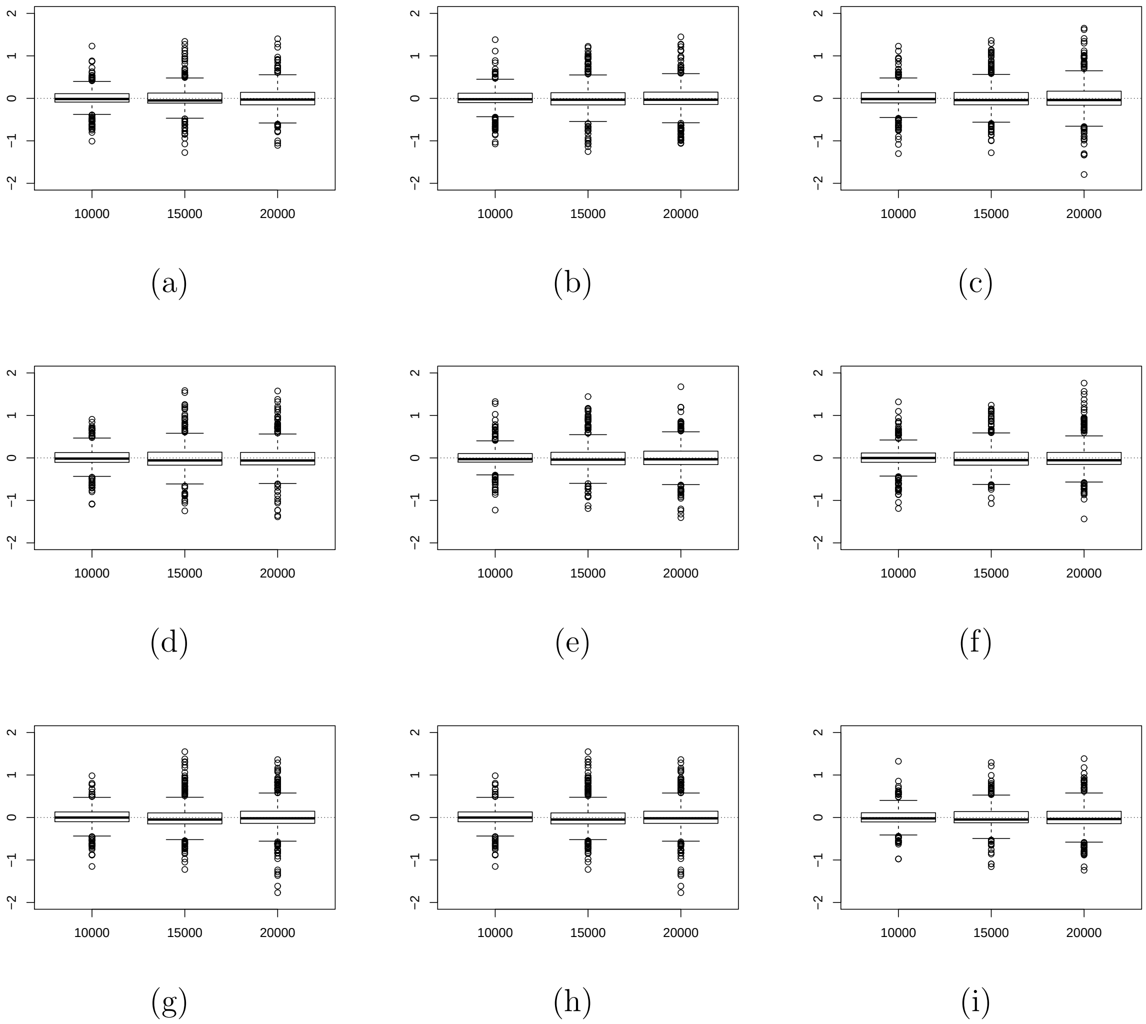


Figure 4: Boxplots of $n^{1/2}\,(\widehat{\boldsymbol{\alpha}}_{\mathrm{MLE}} - \widetilde{\boldsymbol{\alpha}}_{\mathrm{MLE}})$ with $\boldsymbol{\alpha}_0 = (0.133, 0.096, 0.080, 0.079, 0.081, 0.061, 0.056, 0.085, 0.094)^{\mathrm{T}}$ and $n = 10000, 15000, 20000$: (a) $n^{1/2}\,(\hat{\alpha}_{1,\mathrm{MLE}} - \tilde{\alpha}_{1,\mathrm{MLE}})$; (b) $n^{1/2}\,(\hat{\alpha}_{2,\mathrm{MLE}} - \tilde{\alpha}_{2,\mathrm{MLE}})$; (c) $n^{1/2}\,(\hat{\alpha}_{3,\mathrm{MLE}} - \tilde{\alpha}_{3,\mathrm{MLE}})$; (d) $n^{1/2}\,(\hat{\alpha}_{4,\mathrm{MLE}} - \tilde{\alpha}_{4,\mathrm{MLE}})$; (e) $n^{1/2}\,(\hat{\alpha}_{5,\mathrm{MLE}} - \tilde{\alpha}_{5,\mathrm{MLE}})$; (f) $n^{1/2}\,(\hat{\alpha}_{6,\mathrm{MLE}} - \tilde{\alpha}_{6,\mathrm{MLE}})$; (g) $n^{1/2}\,(\hat{\alpha}_{7,\mathrm{MLE}} - \tilde{\alpha}_{7,\mathrm{MLE}})$; (h) $n^{1/2}\,(\hat{\alpha}_{8,\mathrm{MLE}} - \tilde{\alpha}_{8,\mathrm{MLE}})$; (i) $n^{1/2}\,(\hat{\alpha}_{9,\mathrm{MLE}} - \tilde{\alpha}_{9,\mathrm{MLE}})$.

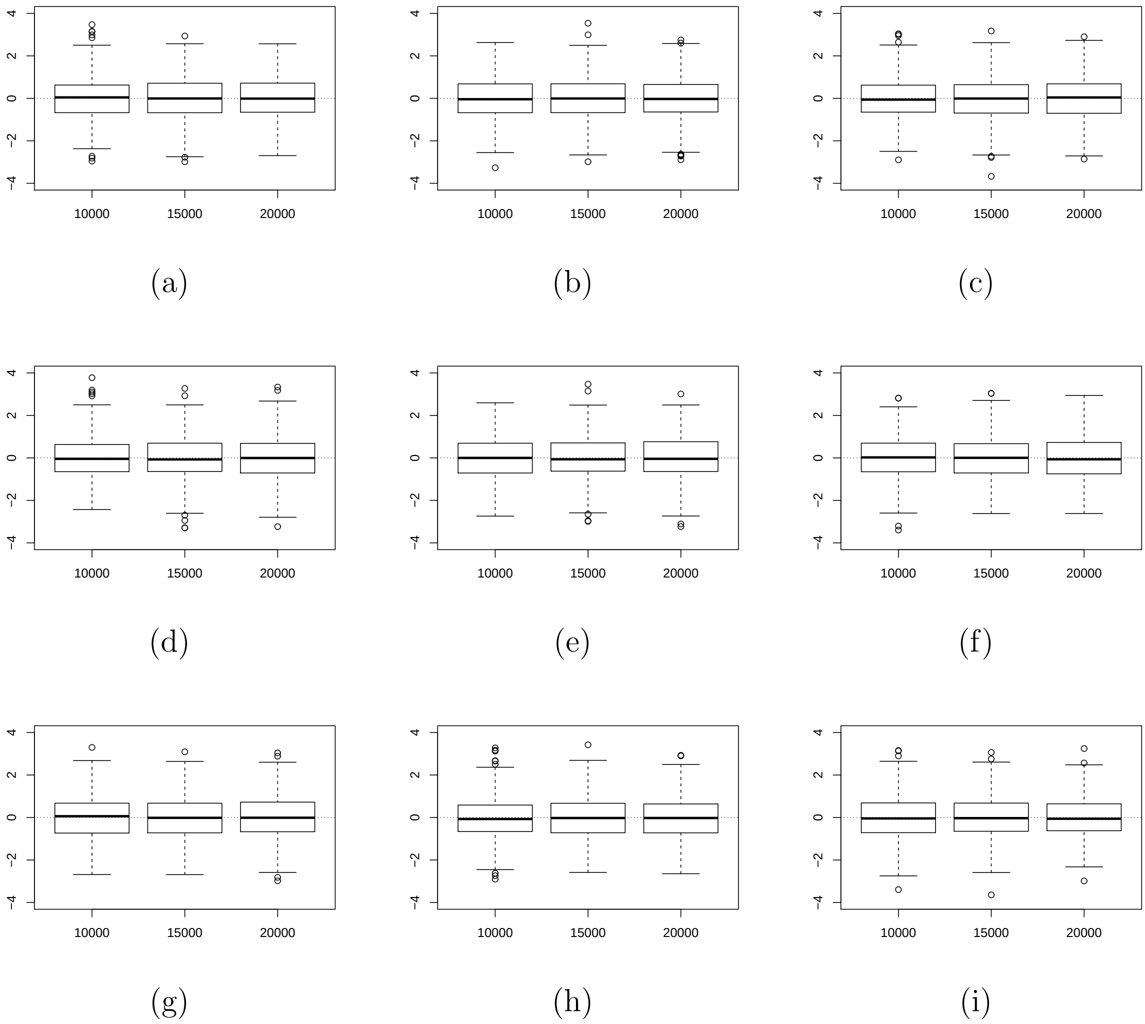


Figure 5: Boxplots of $n^{1/2}\,(\widehat{\boldsymbol{\alpha}}_{\mathrm{LSE}} - \boldsymbol{\alpha}_0)$ with $\boldsymbol{\alpha}_0 = (0.133, 0.096, 0.080, 0.079, 0.081, 0.061, 0.056, 0.085, 0.094)^{\mathrm{T}}$ and $n = 10000, 15000, 20000$: (a) $n^{1/2}\,(\hat{\alpha}_{1,\mathrm{LSE}} - \alpha_{0,1})$; (b) $n^{1/2}\,(\hat{\alpha}_{2,\mathrm{LSE}} - \alpha_{0,2})$; (c) $n^{1/2}\,(\hat{\alpha}_{3,\mathrm{LSE}} - \alpha_{0,3})$; (d) $n^{1/2}\,(\hat{\alpha}_{4,\mathrm{LSE}} - \alpha_{0,4})$; (e) $n^{1/2}\,(\hat{\alpha}_{5,\mathrm{LSE}} - \alpha_{0,5})$; (f) $n^{1/2}\,(\hat{\alpha}_{6,\mathrm{LSE}} - \alpha_{0,6})$; (g) $n^{1/2}\,(\hat{\alpha}_{7,\mathrm{LSE}} - \alpha_{0,7})$; (h) $n^{1/2}\,(\hat{\alpha}_{8,\mathrm{LSE}} - \alpha_{0,8})$; (i) $n^{1/2}\,(\hat{\alpha}_{9,\mathrm{LSE}} - \alpha_{0,9})$.

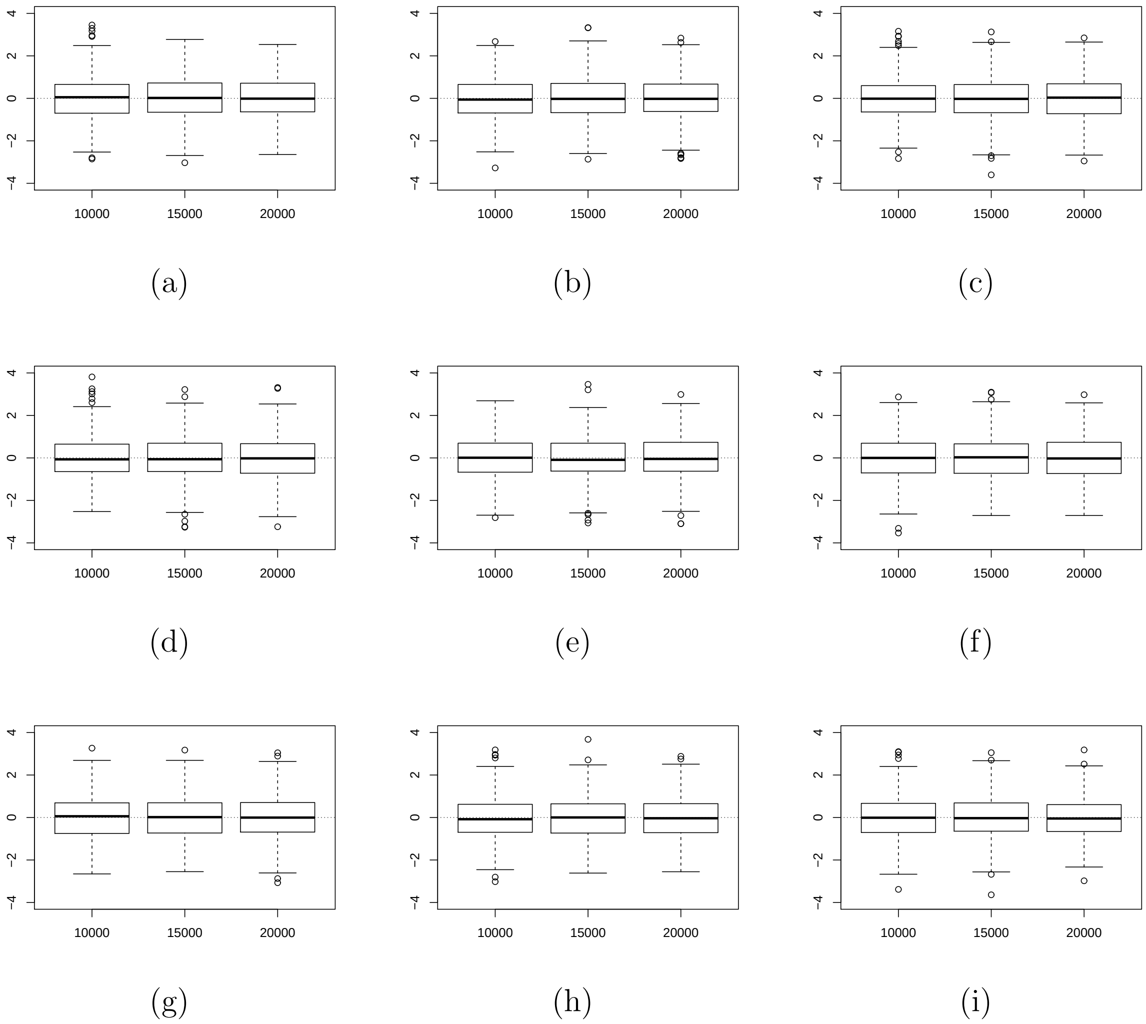
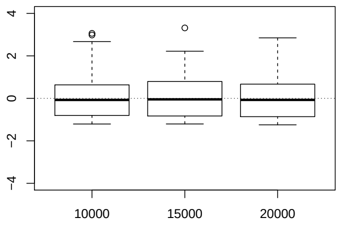
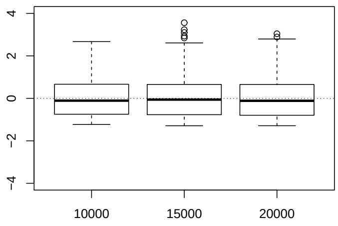


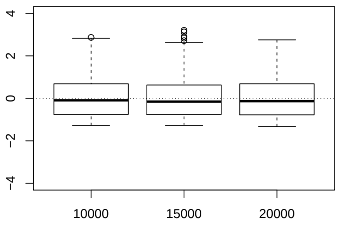
Figure 6: Boxplots of $n^{1/2}\,(\widetilde{\boldsymbol{\alpha}}_{\mathrm{LSE}} - \boldsymbol{\alpha}_0)$ with $\boldsymbol{\alpha}_0 = (0.133, 0.096, 0.080, 0.079, 0.081, 0.061, 0.056, 0.085, 0.094)^{\mathrm{T}}$ and $n = 10000, 15000, 20000$: (a) $n^{1/2}\,(\tilde{\alpha}_{1,\mathrm{LSE}} - \alpha_{0,1})$; (b) $n^{1/2}\,(\tilde{\alpha}_{2,\mathrm{LSE}} - \alpha_{0,2})$; (c) $n^{1/2}\,(\tilde{\alpha}_{3,\mathrm{LSE}} - \alpha_{0,3})$; (d) $n^{1/2}\,(\tilde{\alpha}_{4,\mathrm{LSE}} - \alpha_{0,4})$; (e) $n^{1/2}\,(\tilde{\alpha}_{5,\mathrm{LSE}} - \alpha_{0,5})$; (f) $n^{1/2}\,(\tilde{\alpha}_{6,\mathrm{LSE}} - \alpha_{0,6})$; (g) $n^{1/2}\,(\tilde{\alpha}_{7,\mathrm{LSE}} - \alpha_{0,7})$; (h) $n^{1/2}\,(\tilde{\alpha}_{8,\mathrm{LSE}} - \alpha_{0,8})$; (i) $n^{1/2}\,(\tilde{\alpha}_{9,\mathrm{LSE}} - \alpha_{0,9})$.



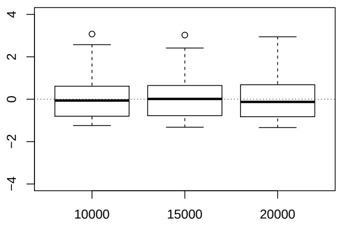
(a)



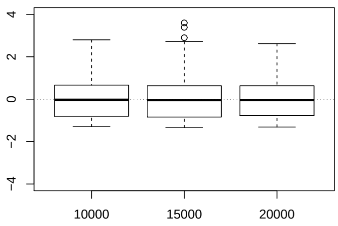
(b)



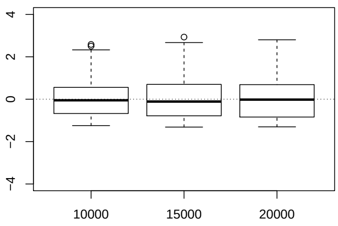
(c)



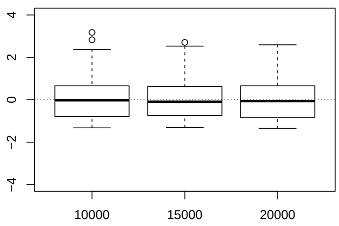
(d)



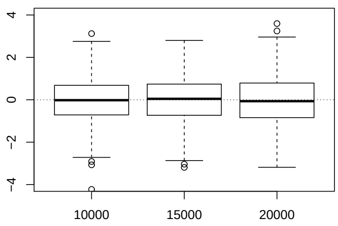
(e)



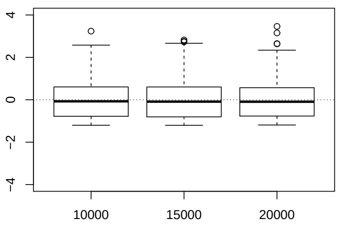
(f)



(g)



(h)



(i)

Figure 7: Boxplots of $n^{1/2}\,(\widehat{\boldsymbol{\alpha}}_{\mathrm{MLE}} - \boldsymbol{\alpha}_0)$ with $\boldsymbol{\alpha}_0 = (0.133, 0.096, 0.080, 0.079, 0.081, 0.061, 0.056, 0.085, 0.094)^{\mathrm{T}}$ and $n = 10000, 15000, 20000$: (a) $n^{1/2}\,(\hat{\alpha}_{1,\mathrm{MLE}} - \alpha_{0,1})$; (b) $n^{1/2}\,(\hat{\alpha}_{2,\mathrm{MLE}} - \alpha_{0,2})$; (c) $n^{1/2}\,(\hat{\alpha}_{3,\mathrm{MLE}} - \alpha_{0,3})$; (d) $n^{1/2}\,(\hat{\alpha}_{4,\mathrm{MLE}} - \alpha_{0,4})$; (e) $n^{1/2}\,(\hat{\alpha}_{5,\mathrm{MLE}} - \alpha_{0,5})$; (f) $n^{1/2}\,(\hat{\alpha}_{6,\mathrm{MLE}} - \alpha_{0,6})$; (g) $n^{1/2}\,(\hat{\alpha}_{7,\mathrm{MLE}} - \alpha_{0,7})$; (h) $n^{1/2}\,(\hat{\alpha}_{8,\mathrm{MLE}} - \alpha_{0,8})$; (i) $n^{1/2}\,(\hat{\alpha}_{9,\mathrm{MLE}} - \alpha_{0,9})$.

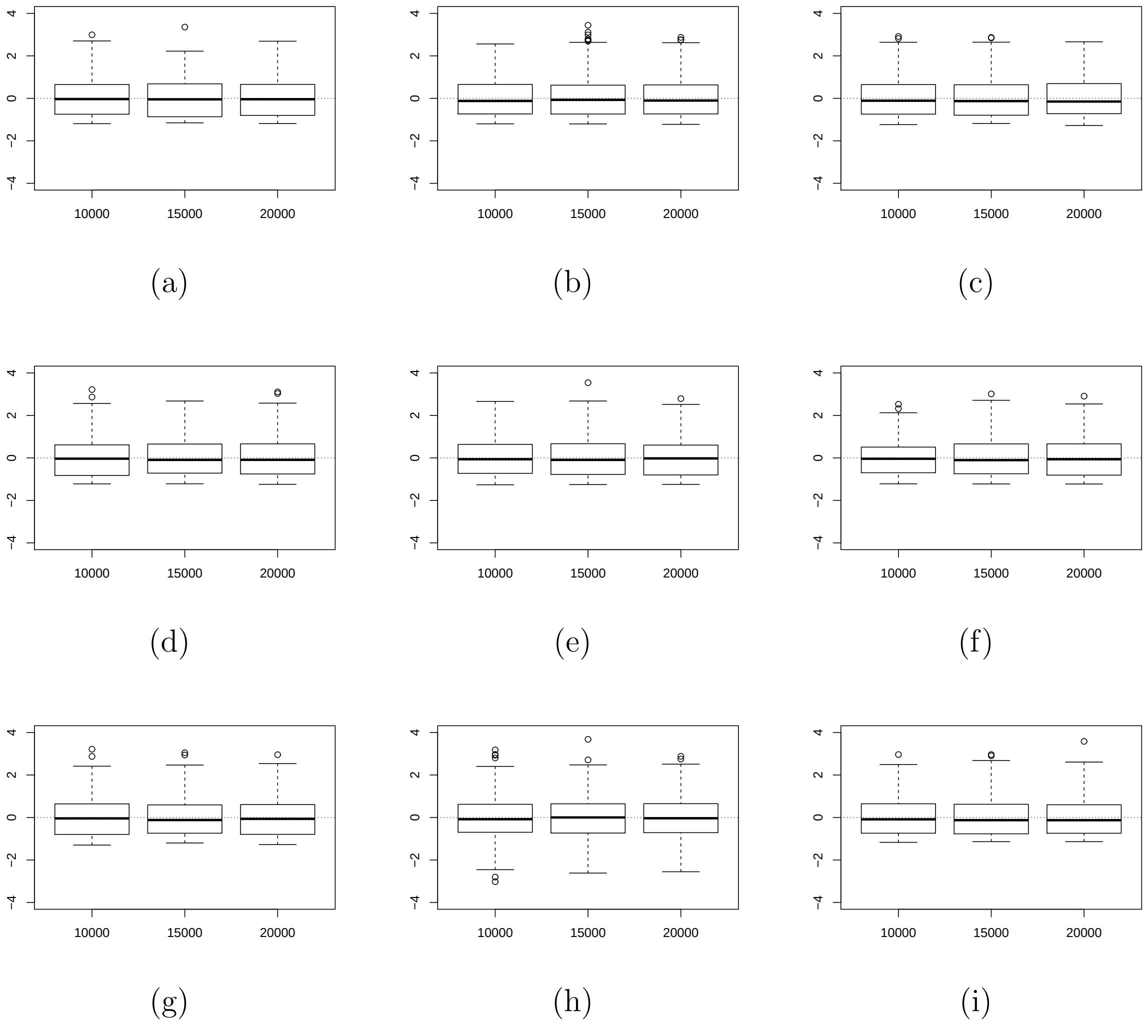


Figure 8: Boxplots of $n^{1/2}\,(\widetilde{\boldsymbol{\alpha}}_{\mathrm{MLE}} - \boldsymbol{\alpha}_0)$ with $\boldsymbol{\alpha}_0 = (0.133, 0.096, 0.080, 0.079, 0.081, 0.061, 0.056, 0.085, 0.094)^{\mathrm{T}}$ and $n = 10000, 15000, 20000$: (a) $n^{1/2}\,(\tilde{\alpha}_{1,\mathrm{MLE}} - \alpha_{0,1})$; (b) $n^{1/2}\,(\tilde{\alpha}_{2,\mathrm{MLE}} - \alpha_{0,2})$; (c) $n^{1/2}\,(\tilde{\alpha}_{3,\mathrm{MLE}} - \alpha_{0,3})$; (d) $n^{1/2}\,(\tilde{\alpha}_{4,\mathrm{MLE}} - \alpha_{0,4})$; (e) $n^{1/2}\,(\tilde{\alpha}_{5,\mathrm{MLE}} - \alpha_{0,5})$; (f) $n^{1/2}\,(\tilde{\alpha}_{6,\mathrm{MLE}} - \alpha_{0,6})$; (g) $n^{1/2}\,(\tilde{\alpha}_{7,\mathrm{MLE}} - \alpha_{0,7})$; (h) $n^{1/2}\,(\tilde{\alpha}_{8,\mathrm{MLE}} - \alpha_{0,8})$; (i) $n^{1/2}\,(\tilde{\alpha}_{9,\mathrm{MLE}} - \alpha_{0,9})$.

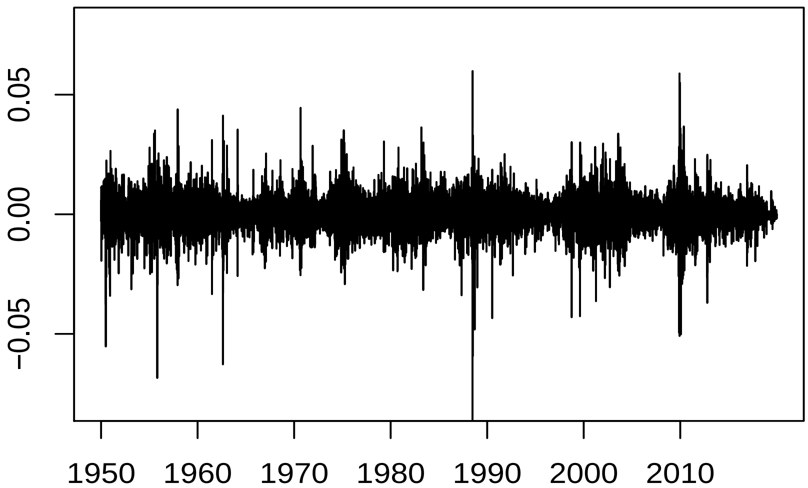


Figure 9: The scatter plot of $\{\hat{x}_t\}_{t=1}^{17276}$ from S&P 500 index daily returns.