Supplement to "A Smooth Simultaneous Confidence Corridor for the Mean of Sparse Functional Data"

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1. Moments of level sets in Lemma A.1

In what follows, $\overline{\Phi} \stackrel{\text{def}}{=} 1 - \Phi$, for any integer a, $a^{[2]} \stackrel{\text{def}}{=} a(a-1)$. For a Gaussian process $X(t), t \in [0,T]$, define the level u upcrossings and downcrossings of $X(t)$ as

$$U_u^X[0,T] = \#\left\{t \in [0,T] : X(t) = u, X'(t) > 0\right\}, \tag{1}$$

$$D_u^X[0,T] = \#\left\{t \in [0,T] : X(t) = u, X'(t) < 0\right\}. \tag{2}$$

Denote $m_t = m(t) = \mathsf{E}\, X(t)$, $r = r(t,s) = \mathrm{Cov}\left\{X(t), X(s)\right\}$, $r_t = r(t,t), r_{0t} = r(0,t), r_{1,0}(t,s) = \frac{\partial}{\partial\alpha} r(\alpha,\beta)\big|_{(t,s)}$ and $r_{1,1}(t,s) = \frac{\partial^2}{\partial\alpha\partial\beta} r(\alpha,\beta)\Big|_{(t,s)}$,

$$M\left\{m(t), m'(t)\right\} = m'(t) + \frac{r_{1,0}(t,t)}{r_t}\left\{u - m(t)\right\}, \tag{3}$$

$$V(t) = r_{1,1}(t,t) - \frac{r_{1,0}^2(t,t)}{r_t}. \tag{4}$$

Additionally, denote the moments of the level sets in (5)-(11) as follows,

$$\begin{aligned} &\mathsf{E}\left(U_u^X[0,T]\right) = G\left\{m(t), m'(t)\right\} \\ &= \int_0^T \frac{\varphi\left\{u - m(t)/\sqrt{r_t}\right\}}{\sqrt{r_t}} \Big(M\left\{m(t), m'(t)\right\} \Phi\left[M\left\{m(t), m'(t)\right\}/\sqrt{V(t)}\right] \\ &\quad + \sqrt{V(t)}\varphi\left[M\left\{m(t), m'(t)\right\}/\sqrt{V(t)}\right]\Big)\, dt, \end{aligned} \tag{5}$$

with $M\left\{m\left(t\right),m'\left(t\right)\right\}$ and $V\left(t\right)$ given in (3) and (4),

$$\begin{aligned}\mathsf{E}\left[U_u^X\left[0,T\right]I_{\{|X(0)|>u\}}\right]&=F\left\{r_{0t},\gamma_1\left(t\right),m\left(0\right),m\left(t\right),m'\left(t\right)\right\}\\&=\int_0^T\varphi\left[\left\{u-m\left(t\right)\right\}/\sqrt{r_t}\right]\left(F_1+F_2+F_3\right)dt,\end{aligned}\tag{6}$$

with F_1,F_2 and F_3 given in (12)-(14),

$$\begin{gathered}\mathsf{E}\left[U_u^X\left[0,T\right]I_{\{|X(0)|\le u\}}\right]\\=G\left\{m\left(t\right),m'\left(t\right)\right\}-F\left\{r_{0t},\gamma_1\left(t\right),m\left(0\right),m\left(t\right),m'\left(t\right)\right\}\\-F\left\{-r_{0t},-\gamma_1\left(t\right),-m\left(0\right),m\left(t\right),m'\left(t\right)\right\},\end{gathered}\tag{7}$$

$$\begin{gathered}\mathsf{E}\left[D_{-u}^X\left[0,T\right]I_{\{|X(0)|\le u\}}\right]\\=G\left\{-m\left(t\right),-m'\left(t\right)\right\}-F\left\{-r_{0t},-\gamma_1\left(t\right),m\left(0\right),-m\left(t\right),-m'\left(t\right)\right\}\\-F\left\{r_{0t},\gamma_1\left(t\right),-m\left(0\right),-m\left(t\right),-m'\left(t\right)\right\},\end{gathered}\tag{8}$$

$$\begin{gathered}\mathsf{E}\left(U_u^X\left[0,T\right]^{[2]}\right)\\=\int_0^T\int_0^T\sigma\left(t,s\right)\sigma\left(s,t\right)J_{11}\left\{b\left(t,s\right),b\left(s,t\right),\rho\right\}\\\frac{\varphi\left(u-m_s/\sqrt{r_s}\right)\varphi\left[\sqrt{r_s/\left(r_sr_t-r^2\right)}\left\{u-m_t-\left(r/r_s\right)\left(u-m_s\right)\right\}\right]}{\sqrt{r_tr_s-r^2}}dsdt,\end{gathered}\tag{9}$$

with $\sigma\left(t,s\right),J_{11}$ and ρ given in (16)-(18),

$$\begin{gathered}\mathsf{E}\left(D_{-u}^X\left[0,T\right]^{[2]}\right)\\=\int_0^T\int_0^T\sigma\left(t,s\right)\sigma\left(s,t\right)J_{11}\left\{-b'\left(t,s\right),-b'\left(s,t\right),\rho\right\}\\\frac{\varphi\left(u+m_s/\sqrt{r_s}\right)\varphi\left[\sqrt{r_s/\left(r_sr_t-r^2\right)}\left\{u+m_t-\left(r/r_s\right)\left(u+m_s\right)\right\}\right]}{\sqrt{r_tr_s-r^2}}dsdt,\end{gathered}\tag{10}$$

$$\begin{gathered}\mathsf{E}\left(U_u^X[0,T]D_{-u}^X[0,T]\right)\\=\int_0^T\int_0^T\sigma\left(t,s\right)\sigma\left(s,t\right)J_{11}\left\{m_1\left(t,s\right),m_2\left(s,t\right),-\rho\right\}\\\frac{\varphi\left(u+m_s/\sqrt{r_s}\right)\varphi\left[\sqrt{r_s/\left(r_sr_t-r^2\right)}\left\{u-m_t+\left(r/r_s\right)\left(u+m_s\right)\right\}\right]}{\sqrt{r_tr_s-r^2}}dsdt,\end{gathered}\tag{11}$$

in which

$$F_1 = \frac{\sqrt{r_t r_{1,1}(t,t) - r_{1,0}^2(t,t)}}{r_t} \varphi \left[\sqrt{1 - \gamma_1^2(t)\left(r_0 r_t - r_{0t}^2\right)\left\{r_t r_{1,1}(t,t) - r_{1,0}^2(t,t)\right\}^{-1}} \right. \tag{12}$$

$$\left. \times \frac{(u - m_t)\left\{\gamma_2(t) + (r_{0t}/r_t)\gamma_1(t)\right\} + m'(t)}{v} \right] \overline{\Phi} \left\{ \sqrt{\left\{r_t r_{1,1}(t,t) - r_{1,0}^2(t,t)\right\} v^{-2} (r_0 r_t - r_{0t}^2)^{-1}} \right.$$

$$\left(u - m_0 - \frac{r_{0t}}{r_t}(u - m_t) + \left(r_0 r_t - r_{0t}^2\right)\gamma_1(t)\left\{r_t r_{1,1}(t,t) - r_{1,0}^2(t,t)\right\} \times \right.$$

$$\left. \left. \left[(u - m_t)\left\{\gamma_2(t) + \frac{r_{0t}}{r_t}\gamma_1(t)\right\} + m'(t) \right] \right) \right\},$$

$$F_2 = \frac{\gamma_1(t)\sqrt{r_0 r_t - r_{0t}^2}}{r_t} \varphi \left[\sqrt{r_t / \left(r_0 r_t - r_{0t}^2\right)} \left\{u - m_0 - r_{0t}(u - m_t)/r_t\right\} \right] \tag{13}$$

$$\times \Phi\left[\left\{m'(t) + \gamma_1(t)(u - m_0) + \gamma_2(t)(u - m_t)\right\} v^{-1}\right],$$

$$F_3 = \frac{\left[m'(t) + (u - m_t)\left\{\gamma_2(t) + (r_{0t}/r_t)\gamma_1(t)\right\}\right]}{\sqrt{r_t}} \tag{14}$$

$$\times \int_{\Phi\left[\sqrt{r_t/\left(r_0 r_t - r_{0t}^2\right)}\left\{u - m_0 - r_{0t}(u - m_t)/r_t\right\}\right]}^{1}$$

$$\Phi\left(\left[m'(t) + \gamma_1(t)\Phi^{-1}(z)\sqrt{r_0 r_t - r_{0t}^2/r_t} + (u - m_t)\left\{\gamma_2(t) + \frac{r_{0t}}{r_t}\gamma_1(t)\right\} \right] v^{-1} \right) dz,$$

$$v^2 = \mathsf{Var}\left\{X'(t) | X(0), X(t)\right\} \tag{15}$$

$$= r_{1,1}(t,t) + \frac{-r_t r_{1,0}^2(t,0) + 2 r_{0t} r_{1,0}(t,t)\, r_{1,0}(t,0) - r_0 r_{1,0}^2(t,t)}{r_0 r_t - r_{0t}^2},$$

$$\sigma^2(t,s) = \mathsf{Var}\left\{X'(t) | X(t), X(s)\right\} \tag{16}$$

$$= r_{1,1}(t,t) - \frac{r_{1,0}(t,t)\left\{r_s r_{1,0}(t,t) - r r_{1,0}(t,s)\right\} + r_{1,0}(t,s)\left\{-r r_{1,0}(t,t) + r_t r_{1,0}(t,s)\right\}}{r_t r_s - r^2},$$

$$J_{11}(a,b,\rho) = (\rho + ab) J_{00} + \sqrt{1 - \rho^2}\varphi(b)\varphi\left\{(-a + \rho b)\left(1 - \rho^2\right)^{-1/2}\right\} \tag{17}$$

$$+ a\varphi(b)\overline{\Phi}\left\{(-a + \rho b)\left(1 - \rho^2\right)^{-1/2}\right\} + b\varphi(a)\overline{\Phi}\left\{(-b + \rho a)\left(1 - \rho^2\right)^{-1/2}\right\},$$

$$\rho = \mathrm{Corr}\left\{X'(t), X'(s) | X(t), X(s)\right\} \tag{18}$$

$$= \frac{r_{1,1}(t,s)}{\sigma(t,s)\sigma(s,t)} - \frac{r_{1,0}(t,t)\left\{r_s r_{1,0}(s,t) - r r_{1,0}(s,s)\right\} + r_{1,0}(t,s)\left\{-r r_{1,0}(s,t) + r_t r_{1,0}(s,s)\right\}}{\sigma(t,s)\sigma(s,t)\left(r_t r_s - r^2\right)},$$

with

$$J_{00} = \Phi\left(a\right)\Phi\left(b\right) + \int_0^{\rho} \frac{1}{2\pi\sqrt{1-z^2}} \exp\left[-\left(a^2+b^2-2zab\right)/\left\{2\left(1-z^2\right)\right\}\right] dz, \tag{19}$$

$$\gamma_1\left(t\right) = \frac{r_{1,0}(t,0)r_t - r_{1,0}\left(t,t\right)r_{0t}}{r_0 r_t - r_{0t}^2},\ \gamma_2\left(t\right) = \frac{r_{1,0}(t,t)r_0 - r_{1,0}(t,0)r_{0t}}{r_0 r_t - r_{0t}^2}, \tag{20}$$

$$b(t,s) = \frac{m'(t)}{\sigma(t,s)} + (u-m_t)\frac{r_s r_{1,0}(t,t) - r r_{1,0}(t,s)}{\sigma(t,s)\left(r_t r_s - r^2\right)} + (u-m_s)\frac{r_t r_{1,0}(t,s) - r r_{1,0}(t,t)}{\sigma(t,s)\left(r_t r_s - r^2\right)}, \tag{21}$$

$$b'(t,s) = \frac{m'(t)}{\sigma(t,s)} - (u+m_t)\frac{r_s r_{1,0}(t,t) - r r_{1,0}(t,s)}{\sigma(t,s)\left(r_t r_s - r^2\right)} - (u+m_s)\frac{r_t r_{1,0}(t,s) - r r_{1,0}(t,t)}{\sigma(t,s)\left(r_t r_s - r^2\right)}, \tag{22}$$

$$m_1(t,s) = \frac{m'(t)}{\sigma(t,s)} + (u-m_t)\frac{r_s r_{1,0}(t,t) - r r_{1,0}(t,s)}{\sigma(t,s)\left(r_t r_s - r^2\right)} - (u+m_s)\frac{r_t r_{1,0}(t,s) - r r_{1,0}(t,t)}{\sigma(t,s)\left(r_t r_s - r^2\right)}, \tag{23}$$

$$m_2(s,t) = -\frac{m'(t)}{\sigma(s,t)} - (u-m_t)\frac{r_s r_{1,0}(s,t) - r r_{1,0}(s,s)}{\sigma(s,t)\left(r_t r_s - r^2\right)} + (u+m_s)\frac{r_t r_{1,0}(s,s) - r r_{1,0}(s,t)}{\sigma(t,s)\left(r_t r_s - r^2\right)}. \tag{24}$$

2. Proof of Lemma A.5

Lemma A.5 *Under Assumptions (A5)-(A6), for* $x \in [0,1]$

$$0 < D_0\left(K\right) \leq D_x\left(K\right) \leq D_{1/2}\left(K\right) = \mu_2\left(K\right) < +\infty, \tag{25}$$

while $\sup_{x\in[0,1]}|C_x(K)| < \infty$.

Proof. It is obvious that $D_x\left(K\right) = \mu_2\left(K\right) > 0, \forall x \in [h, 1-h]$ while for $x \in [0,h)$,

$$D_x\left(K\right) = \frac{1}{2}\int_{-x/h}^{1} u^2 K\left(u\right)du \int_{-x/h}^{1} K\left(v\right)dv +$$
$$\frac{1}{2}\int_{-x/h}^{1} K\left(u\right)du \int_{-x/h}^{1} v^2 K\left(v\right)dv - \int_{-x/h}^{1} uK\left(u\right)du \int_{-x/h}^{1} vK\left(v\right)dv =$$
$$\int_{-x/h}^{1}\int_{-x/h}^{1} \frac{\left(u-v\right)^2}{2} K\left(u\right)K\left(v\right)dudv \geq \int_0^1\int_0^1 \frac{\left(u-v\right)^2}{2} K\left(u\right)K\left(v\right)dudv = D_0\left(K\right)$$

hence $D_x\left(K\right) \geq D_0\left(K\right) > 0$,for $x \in [0,h)$ and similarly for $x \in [1-h,1]$. The upper bound for $D_x\left(K\right)$ follows likewise. To bound $C_x(K)$, we show that $\mu_{0,x}\{K_x^*\left(x\right)\}^2 \geq C > 0$ for a constant C. The result is obvious if $x \in [h, 1-h]$. For $x \in [0,h)$, applying the upper bound from (25)

$$\mu_{0,x}\{K_x^*\left(x\right)^2\} = \int_{-x/h}^{1} K^2\left(u\right)\left\{\mu_{2,x}\left(K\right) - \mu_{1,x}\left(K\right)u\right\}^2 D_x^{-2}\left(K\right)du$$
$$\geq \mu_2^{-2}\left(K\right)\int_0^1 K^2\left(u\right)\left\{\mu_{2,x}\left(K\right) - \mu_{1,x}\left(K\right)u\right\}^2 du.$$

Note that for $u \in [0,1], x \in [0,h)$

$$\left\{\mu_{2,x}(K) - \mu_{1,x}(K)u\right\}^2 = \{\int_{-x/h}^{1} (u-v)vK(v)dv\}^2$$
$$\geq \min[\{\int_{0}^{1} (u-v)vK(v)dv\}^2, \{\int_{-1}^{1} (u-v)vK(v)dv\}^2]$$

hence for $x \in [0,h)$

$$\mu_{0,x}\{K_x^*(x)\}^2 \geq \mu_2^{-2}(K) \times$$
$$\min[\int_0^1 K^2(u)\{\int_0^1 (u-v)vK(v)dv\}^2 du, \int_0^1 K^2(u)\{\int_{-1}^1 (u-v)vK(v)dv\}^2 du] > 0.$$

The lemma is proved.

3. Proof of Lemma A.6

Lemma A.6 Under Assumptions (A1)-(A6), for $D_x(K)$ given in (3.2) and $\mathbf{D}_x$ in (A.3),

$$\left(\mathbf{X}^{\mathsf{T}}\mathbf{W}\mathbf{X}\right)^{-1} = f^{-1}(x)\operatorname{diag}\left(1, h^{-1}\right)\left\{D_x^{-1}(K)\mathbf{D}_x + \Delta_{1,n}(x)\right\}\operatorname{diag}\left(1, h^{-1}\right)$$

as $n \to \infty$, where the 2×2 random matrices $\Delta_{1,n}(x) = \mathcal{U}(h) + \mathcal{U}_{a.s.}\{\sqrt{\log n/(nh)}\}$.

Proof. For notational simplicity, let $x \in [h, 1-h]$, we investigate $s_{n,l}(x), l = 0,1,2$, given in (2.7).

$$|s_{n,0}(x) - f(x)| \leq \left|n(\mathsf{E}\,N_1)N_{\mathrm{T}}^{-1} - 1\right| \left|(n\,\mathsf{E}\,N_1)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{N_i} K_h(X_{ij}-x)\right| + \tag{26}$$
$$|\mathsf{E}\,K_h(X_{ij}-x) - f(x)| + \left|(n\,\mathsf{E}\,N_1)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{N_i} K_h(X_{ij}-x) - \mathsf{E}\,K_h(X_{ij}-x)\right|$$
$$= I_1(x) + I_2(x) + I_3(x).$$

Clearly, $I_2(x) = \mathcal{U}(h^2)$ and $\mathsf{E}\{K_h(X_{ij}-x)\}^r = \mathcal{U}(h^{1-r})$ for $r \geq 2$. Define $I_3(x) = (n\,\mathsf{E}\,N_1)^{-1}|\sum_{i=1}^n \zeta_{i,h}|$ with $\zeta_{i,h} = \sum_{j=1}^{N_i} K_h(X_{ij}-x) - \mathsf{E}\,K_h(X_{ij}-x)\,\mathsf{E}\,N_1$. For large n,

$$\mathsf{E}\,|\zeta_{i,h}|^r = \mathsf{E}\left|\sum_{j=1}^{N_i} K_h(X_{ij}-x) - \mathsf{E}\,K_h(X_{ij}-x)\,\mathsf{E}\,N_1\right|^r \leq \tag{27}$$
$$2^{r-1}[\mathsf{E}\{\sum_{j=1}^{N_i} K_h(X_{ij}-x)\}^r + \{\mathsf{E}\,K_h(X_{ij}-x)\,\mathsf{E}\,N_1\}^r] \leq 2^r\,\mathsf{E}\{\sum_{j=1}^{N_i} K_h(X_{ij}-x)\}^r =$$
$$2^r\,\mathsf{E}\left[\sum_{0\leq r_1,\ldots,r_{N_i}\leq r}^{r_1+\ldots+r_{N_i}=r}\binom{r}{r_1\ldots r_{N_i}}\prod_{i=1}^{N_i}\mathsf{E}\{K_h(X_{ij}-x)\}^{r_i}\right] \leq 2^r\,\mathsf{E}\left[N_i^r \max_{0\leq r_1,\ldots,\,r_{N_i}\leq r}^{r_1+\ldots+r_{N_i}=r}\prod_{i=1}^{N_i}\mathsf{E}\{K_h(X_{ij}-x)\}^{r_i}\right]$$
$$\leq 2^r\left(\mathsf{E}\,N_1^r\right)C_{K,f}^r h^{1-r} \leq 2^r\left(c_N^r r!\right)C_{K,f}^r h^{1-r} = \left(2c_N C_{K,f}\right)^r r! h^{1-r} = C_\zeta r! h^{1-r}.$$

It can be next verified that $\mathsf{E}\left(\zeta_{i,h}\right)^2 = (\mathsf{E}\, N_1)\, h^{-1} f\,(x) \int K^2\,(v)\,dv\,\{1+u(1)\}$. Hence, $\exists\ C'_\zeta > c'_\zeta > 0$ such that $c'_\zeta h^{-1} < \mathsf{E}\left(\zeta_{i,h}\right)^2 < C'_\zeta h^{-1}$, i.e., $\mathsf{E}\,|\zeta_{i,h}|^r \le c_*^{r-2} r!\,\mathsf{E}\left(\zeta_{i,h}\right)^2$ with $c_* = (C_\zeta/c'_\zeta)^{\frac{1}{r-2}} h^{-1}$, see (27). In fact, it implies that $\left\{\zeta_{i,h}\right\}_{i=1}^n$ satisfies Cramér's Condition. Therefore, applying Lemma A.4 to $\sum_{i=1}^n \zeta_{i,h}$, for large n and large $\delta > 0$, one shows

$$
\mathrm{P}\{I_3\,(x) > \delta\sqrt{\log n/\,(nh)}\} \le
$$
$$
2\exp[-\left(\mathsf{E}\,N_1\right)^2 \delta^2 \log n\{4C'_\zeta + 2\delta\,\mathsf{E}\,N_1 \left(C_\zeta/c'_\zeta\right)^{1/(r-2)} \sqrt{\log n/\,(nh)}\}^{-1}] \le 2n^{-C\delta^2} \le 2n^{-8}.
$$

Now discretize $h = x_0 < x_1 < \cdots < x_{M_n} = 1-h$ with $M_n = n^4$ and then,

$$
\mathrm{P}\{\max_{j=0}^{M_n} I_3\,(x_j) > \delta\sqrt{\log n/\,(nh)}\} \le \sum_{j=0}^{M_n} \mathrm{P}\{|I_3\,(x)| > \delta\sqrt{\log n/\,(nh)}\} \le 2n^{-4},
$$

and hence the Borel-Cantelli Lemma implies that $\max_{j=0}^{M_n} I_3\,(x_j) = \mathcal{O}_{a.s.}\{\sqrt{\log n/(nh)}\}$. Also clearly,

$$
\begin{aligned}
\sup_{x\in[h,1-h]} I_3\,(x) &\le \max_{j=0}^{M_n} I_3\,(x_j) + \max_{j=0}^{M_n-1} \sup_{x\in[x_j,x_{j+1}]} |I_3\,(x_j) - I_3\,(x)| \\
&\le \mathcal{O}_{a.s.}\{\sqrt{\log n/(nh)}\} + \mathcal{U}\left\{(1-2h)\,/\left(nh^4\right)\right\} = \mathcal{O}_{a.s.}\{\sqrt{\log n/(nh)}\},
\end{aligned}
$$

which by the definition of $I_3\,(x)$ implies that

$$
\begin{aligned}
(n\,\mathsf{E}\,N_1)^{-1} \sum_{i=1}^n \sum_{j=1}^{N_i} K_h(X_{ij}-x) &= \mathsf{E}\,K_h(X_{ij}-x) + \mathcal{U}_{a.s.}\{\sqrt{\log n/(nh)}\} \\
&= f(x) + U(h^2) + \mathcal{U}_{a.s.}\{\sqrt{\log n/(nh)}\}.
\end{aligned} \tag{28}
$$

Applying Lemma A.4 for N_T, one has $|(n\,\mathsf{E}\,N_1)\,/N_\mathrm{T} - 1| = \mathcal{O}_{a.s}\{\sqrt{\log n/n}\}$ and (28) also guarantees that $\sup_{x\in[h,1-h]} I_1(x) = \mathcal{O}_{a.s.}(\sqrt{\log n/n})$. Now, by (26), $s_{n,0}\,(x) = f\,(x) + \mathcal{U}\left(h^2\right) + \mathcal{U}_{a.s.}\{\sqrt{\log n/(nh)}\}$. Similarly, $s_{n,1}\,(x) = \mathcal{U}\,(h) + \mathcal{U}_{a.s.}\{\sqrt{\log n/(nh)}\ \}$and $s_{n,2}(x) = f(x)\mu_2(K) + \mathcal{U}\,(h) + \mathcal{U}_{a.s.}\{\sqrt{\log n/(nh)}\}$ which imply that $\mathbf{X}^\mathsf{T}\mathbf{W}\mathbf{X}$ can be written as

$$
f\,(x)\,\mathrm{diag}(1,h)[\mathrm{diag}\,\{1,\mu_2(K)\} + \mathcal{U}\,(h) + \mathcal{U}_{a.s.}\{\sqrt{\log n/(nh)}\}]\,\mathrm{diag}(1,h).
$$

Finally, the inverse of this matrix is concluded as this lemma.

4. Proof of Lemma A.8

Lemma A.8 Under Assumptions (A1)-(A6), for $\widehat{\varepsilon}(x)$ and $\widehat{\xi}_k(x)$ given in (A.4),

$$\widetilde{e}(x) = \{1 + \Delta_{2,n}(x)\}\{\widehat{\varepsilon}(x) + \widehat{\xi}(x)\}$$

as $n \to \infty$, where the 2×2 random matrices $\Delta_{2,n}(x) = \mathcal{U}(h) + \mathcal{U}_{a.s.}\{\sqrt{\log n/(nh)}\}$.

Proof. For notational simplicity, let $x \in [h, 1-h]$, therefore $\widehat{\varepsilon}(x) + \widehat{\xi}(x) = f^{-1}(x)T_0(x)$ with $T_l, l = 0, 1$ defined as

$$T_l(x) = N_{\mathrm{T}}^{-1}\sum\nolimits_{i,j} K_h(X_{ij} - x)\{(X_{ij} - x)/h\}^l\{\sigma(X_{ij})\varepsilon_{ij} + \sum\nolimits_{k=1}^{\infty}\phi_k(X_{ij})\xi_{ik}\}.$$

Lemma A.6 shows that for $\Delta_{1,n}(x)$ given in Lemma A.6

$$\widetilde{e}(x) = f^{-1}(x)e_0^{\mathsf{T}}\operatorname{diag}\left(1, h^{-1}\right)\left[\operatorname{diag}\left\{1, \mu_2^{-1}(K)\right\} + \Delta_{1,n}(x)\right]\{T_0(x), T_1(x)\}^{\mathsf{T}},$$

i.e., $\widetilde{e}(x) = \{1 + \Delta_{1,n}(x)\}f^{-1}(x)T_0(x)$. Therefore, this lemma holds.

5. Proof of Lemma A.9

Lemma 1 *Given (A1)-(A6), then there exists a sequence of Wiener processes* $\{W_{N_T}(t)\}_{t=1}^{N_T}$ *independent of* $\{N_i,\ X_{ij},\ \xi_i\ 1 \le i \le n,\ 1 \le j \le N_i,\ 1 \le k \le \infty\}$ *such that as* $n \to \infty$ *and for some* $t' > 2/5$

$$\left\|\widehat{\varepsilon}(x) - \widehat{\varepsilon}_{N_T}(x)\right\|_{\infty} = \mathcal{O}_{a.s.}(n^{-t'}),$$

with $\widehat{\varepsilon}_{N_T}(x) = \{N_T f(x)\}^{-1}\sum_{t=1}^{N_T} K^*_{x,h}\left(X_{(t)} - x\right)\sigma\left(X_{(t)}\right)\{W_{N_T}(t) - W_{N_T}(t-1)\}$.

Proof. Without loss of generality, let $x \in [h, 1-h]$. By Lemma A.3, let $H(x) = x^r, r > 5$ (Assumption A4) and $x_n = n^s, s \in \left(2r^{-1}, 2/5\right)$. It is easy to verify that $\left\{\varepsilon_{(t)}\right\}_{t=1}^{N_{\mathrm{T}}}$ satisfies the conditions of Lemma A.3 and $nH^{-1}(ax_n) = a^{-r}n^{1-rs} = \mathcal{O}(n^{-s'})$ for some $s' > 1$. Therefore, there exists a sequence of Wiener process $\{W_{N_{\mathrm{T}}}(t)\}_{t=1}^{N_{\mathrm{T}}}$ independent of $\{N_i,\ X_{ij},\ \xi_i\ 1 \le i \le n,\ 1 \le j \le N_i,\ 1 \le k \le \infty\}$ such that $P\{M_{N_{\mathrm{T}}} > n^s\} \le C_2 n^{-s'}$ with $M_{N_{\mathrm{T}}} = \max_{1 \le q \le N_{\mathrm{T}}}|S_q - W_{N_{\mathrm{T}}}(q)|$ and hence Borel-Cantelli Lemma warrants that $M_{N_{\mathrm{T}}} = \mathcal{O}_{a.s}(n^s)$.

The technique of summation by parts implies that

$$\sup_{x \in [h,1-h]}|\widehat{\varepsilon}(x) - \widehat{\varepsilon}_{N_{\mathrm{T}}}(x)| \le \sup_{x \in [h,1-h]} N_{\mathrm{T}}^{-1}c_f^{-1}|K_h\left(X_{(N_{\mathrm{T}})} - x\right)\sigma\left(X_{(N_{\mathrm{T}})}\right)\{W_{N_{\mathrm{T}}}(N_{\mathrm{T}}) - S_{N_{\mathrm{T}}}\} \tag{29}$$
$$+ \sum\nolimits_{t=1}^{N_{\mathrm{T}}-1}\{K_h\left(X_{(t)} - x\right)\sigma\left(X_{(t)}\right) - K_h\left(X_{(t+1)} - x\right)\sigma\left(X_{(t+1)}\right)\}\{W_{N_{\mathrm{T}}}(t) - S_t\}| \le h^{-1}M_{N_{\mathrm{T}}}N_{\mathrm{T}}^{-1}c_f^{-1}\times$$
$$\sup_{x \in [h,1-h]}[\ 3C_K C_\sigma + \sum\nolimits_{1 \le t \le N_{\mathrm{T}}-1}^{X_{(t)} \in [x-h, x+h]}|K\{\left(X_{(t)} - x\right)/h\}\sigma\left(X_{(t)}\right) - K\{\left(X_{(t+1)} - x\right)/h\}\sigma\left(X_{(t+1)}\right)|\ ].$$

Since $|ab-cd| \leq |a||b-d| + |b||a-c| + |a-c||b-d|$, (29) is bounded by

$$h^{-1} M_{N_{\mathrm{T}}} N_{\mathrm{T}}^{-1} c_f^{-1} \sup_{x\in[h,1-h]} [\ 3C_K C_\sigma + \sum\nolimits_{1\leq t\leq N_{\mathrm{T}}-1}^{X_{(t)}\in[x-h,x+h]} 2C_K \times$$
$$|\sigma\left(X_{(t)}\right) - \sigma\left(X_{(t+1)}\right)| + C_\sigma |K\{\left(X_{(t)}-x\right)/h\} - K\{\left(X_{(t+1)}-x\right)/h\}|\ ].$$

Therefore, $\exists$ constants $L^1_{K,\sigma}, L^2_{K,\sigma}, C$ and C' such that (29) is bounded by

$$h^{-1} M_{N_{\mathrm{T}}} N_{\mathrm{T}}^{-1} c_f^{-1} \sup_{x\in[h,1-h]} (\ 3C_K C_\sigma + L^1_{K,\sigma} \sum\nolimits_{1\leq t\leq N_{\mathrm{T}}-1}^{X_{(t)}\in[x-h,x+h]} |X_{(t)} - X_{(t+1)}| +$$
$$L^2_{K,\sigma} h^{-1} \sum\nolimits_{1\leq t\leq N_{\mathrm{T}}-1}^{X_{(t)}\in[x-h,x+h]} |X_{(t)} - X_{(t+1)}|\ ) \leq h^{-1} M_{N_{\mathrm{T}}} N_{\mathrm{T}}^{-1} \left(C + C'h\right).$$

Namely $\sup_{x\in[h,1-h]} |\widehat{\varepsilon}(x) - \widehat{\varepsilon}_{N_{\mathrm{T}}}(x)| = \mathcal{O}_{a.s}(h^{-1}n^{s-1})$ and by assumption (A5), one obtains

$$\sup_{x\in[h,1-h]} |\widehat{\varepsilon}_{N_{\mathrm{T}}}(x) - \widehat{\varepsilon}(x)| = \mathcal{O}_{a.s.}(n^{-t'}),\ t' > 2/5.$$

This completes the proof.

6. Proof of Lemma A.10

Lemma A.10 Under Assumptions (A1)-(A6), as $n\to\infty$,

$$\left\| N_{\mathrm{T}}^{-1} \sum\nolimits_{i,j} R^2_{ij,\varepsilon}(x) - \mathsf{E}\, R^2_{1,\varepsilon}(x) \right\|_\infty = \mathcal{O}_{a.s.}\{\sqrt{\log n/(nh)}\},$$

$$\left\| N_{\mathrm{T}}^{-1} \sum\nolimits_{i=1}^n R^2_{i,\xi}(x) - (\mathsf{E}\, N_1)^{-1} \mathsf{E}\, R^2_{1,\xi}(x) \right\|_\infty = \mathcal{O}_{a.s.}\{\sqrt{\log n/(nh)}\},$$

with $R_{ij,\varepsilon}(x)$ and $R_{ik,\xi_k}(x)$ given in (A.5).

Proof. Without loss of generality, let $x \in [h, 1-h]$. Clearly,

$$\sup_{x\in[h,1-h]} \left| N_{\mathrm{T}}^{-1} \sum\nolimits_{i=1}^n R^2_{ik,\xi}(x) - (\mathsf{E}\, N_1)^{-1} \mathsf{E}\, R^2_{i,\xi}(x) \right| \leq (\mathsf{E}\, N_1)^{-1} \sup_{x\in[h,1-h]} \left| n^{-1} \sum\nolimits_{i=1}^n R^2_{i,\xi}(x) - \mathsf{E}\, R^2_{i,\xi}(x) \right|$$
$$+ (\mathsf{E}\, N_1)^{-1} \sup_{x\in[h,1-h]} \left| n(\mathsf{E}\, N_1) N_{\mathrm{T}}^{-1} - 1 \right| \left| n^{-1} \sum\nolimits_{i=1}^n R^2_{i,\xi}(x) \right|.$$

Next, it is straightforward to verify the Cramér's Condition for $R^2_{i,\xi}(x)^* = R^2_{i,\xi}(x) - \mathsf{E}\, R^2_{i,\xi}(x)$, i.e., under $r\geq 2$ and $c_* \sim h^{-1}$, $\mathsf{E}\{R^2_{i,\xi}(x)^*\}^r \leq c_*^{r-2} r!\, \mathsf{E}\, R^2_{i,\xi}(x)^*$, so $\sup_{x\in[h,1-h]} \left| n^{-1} \sum_{i=1}^n R^2_{i,\xi}(x) - \mathsf{E}\, R^2_{i,\xi}(x) \right| = \mathcal{O}_{a.s.}\{\sqrt{\log n/(nh)}\}$, i.e., $n^{-1}\sum_{i=1}^n R^2_{i,\xi}(x) = \mathcal{U}\left(h^{-1}\right) + \mathcal{U}_{a.s.}\{\sqrt{\log n/(nh)}\}$. Therefore,

$$\sup_{x\in[h,1-h]} \left| N_{\mathrm{T}}^{-1} \sum\nolimits_{i=1}^n R^2_{i,\xi}(x) - (\mathsf{E}\, N_1)^{-1} \mathsf{E}\, R^2_{i,\xi}(x) \right| = \mathcal{O}_{a.s.}\{\sqrt{\log n/(nh)}\}.$$

The proof for $R^2_{ij,\varepsilon}(x)$ is similar.

7. Proof of Lemma A.11

Lemma A.11 *Under Assumptions (A1)-(A6)*

$$\lim_{n\to\infty}\sup_{t\in[0,h^{-1}]}|r_{1,1}(t,t)-C(t)|=0. \tag{30}$$

There exist constants $0<c<C<\infty$, $1>\delta>0$, *such that for large* n

$$\inf_{t,s\in[0,h^{-1}],|t-s|<2} r(t,s)\geq -1+c>-1, \quad \sup_{2>|t-s|\geq\delta,\ t,s\in[0,h^{-1}]} r(t,s)\leq 1-c<1, \tag{31}$$

$$\sup_{0<|t-s|<\delta,t,s\in[0,h^{-1}]} \max[r_{1,0}(t,s)/(t-s),\{1-r^2(t,s)\}/(t-s)^2]\leq C,$$
$$\inf_{0<|t-s|<\delta,t,s\in[0,h^{-1}]} \min[r_{1,0}(t,s)/(t-s),\{1-r^2(t,s)\}/(t-s)^2]\geq c, \tag{32}$$

$$\sup_{0<|t-s|<\delta,t,s\in[0,h^{-1}]} \frac{r_{1,1}(t,t)-r^2_{1,0}(t,s)/\left(1-r^2\right)}{(t-s)^2}\leq C,$$
$$\inf_{0<|t-s|<\delta,t,s\in[0,h^{-1}]} \frac{r_{1,1}(t,t)-r^2_{1,0}(t,s)/\left(1-r^2\right)}{(t-s)^2}\geq c, \tag{33}$$

$$\sup_{|t-s|<2,t,s\in[0,h^{-1}]} |r^2_{1,0}(t,s)/\{1-r^2(t,s)\}|\leq C \tag{34}$$

$$\inf_{|t-s|<2,t,s\in[0,h^{-1}]} \frac{|r_{1,0}(t,s)/(1+r)|}{\sqrt{r_{1,1}(t,t)-r^2_{1,0}(t,s)/(1-r^2)}}\geq c \tag{35}$$

Proof. Let $t\in\left[1,h^{-1}-1\right]$ for notational simplicity. By definition, $r(t,s)=\mathsf{E}\,R(t,s)$ with

$$R(t,s)=\left[\sum\nolimits_{k,i}\left\{\sum\nolimits_j K(X_{ij}/h-t)\phi_k(X_{ij})\right\}\left\{\sum\nolimits_j K(X_{ij}/h-s)\phi_k(X_{ij})\right\}+\sum\nolimits_{i,j}\sigma^2(X_{ij})\times\right. \tag{36}$$
$$\left.K(X_{ij}/h-t)K(X_{ij}/h-s)\right]\times\left[\sum\nolimits_{k,i}\left\{\sum\nolimits_j K(X_{ij}/h-t)\phi_k(X_{ij})\right\}^2+\sum\nolimits_{i,j}K^2(X_{ij}/h-t)\sigma^2(X_{ij})\right]^{-1/2}$$
$$\times\left[\sum\nolimits_{k,i}\left\{\sum\nolimits_j K(X_{ij}/h-s)\phi_k(X_{ij})\right\}^2+\sum\nolimits_{i,j}K^2(X_{ij}/h-s)\sigma^2(X_{ij})\right]^{-1/2}.$$

Now let $R_{1,1}(t,t)=\partial R(t,s)/\partial t\partial s|_{t=s}$, $K'(x)=dK(x)/dx$ and simple algebra shows that

$$R_{1,1}(t,t)=\left\{I_1(t)I_2(t)-I_3^2(t)\right\}I_2^{-2}(t), \tag{37}$$

in which

$$I_1(t) = \sum\nolimits_{k,i} \left\{ \sum\nolimits_j K'(X_{ij}/h - t) \phi_k(X_{ij}) \right\}^2 + \sum\nolimits_{i,j} \left\{ K'\left(X_{ij}h^{-1} - t\right) \right\}^2 \sigma^2(X_{ij}),$$

$$I_2(t) = \sum\nolimits_{k,i} \left\{ \sum\nolimits_j K(X_{ij}/h - t) \phi_k(X_{ij}) \right\}^2 + \sum\nolimits_{i,j} K^2(X_{ij}/h - t) \sigma^2(X_{ij}),$$

$$I_3(t) = \sum\nolimits_{k,i} \left\{ \sum\nolimits_j K'(X_{ij}/h - t) \phi_k(X_{ij}) \right\} \times$$
$$\left\{ \sum\nolimits_j K(X_{ij}/h - t) \phi_k(X_{ij}) \right\} + \sum\nolimits_{i,j} \sigma^2(X_{ij}) K(X_{ij}/h - t) K'(X_{ij}/h - t).$$

Now it is easy to verify that

$$R_{1,1}(t,t) \leq \left[\sum\nolimits_{k,i} N_i \sum\nolimits_j \left\{ K'\left(X_{ij}h^{-1} - t\right) \right\}^2 \phi_k^2(X_{ij}) + \sum\nolimits_{i,j} \left\{ K'\left(X_{ij}h^{-1} - t\right) \right\}^2 \sigma^2(X_{ij}) \right]$$
$$\times \left\{ \sum\nolimits_{i,j} K^2\left(X_{ij}h^{-1} - t\right) \sigma^2(X_{ij}) \right\}^{-1}$$
$$\leq \kappa (C_\phi C_{K'})^2 (c_\sigma c_K)^{-2} N_{\mathrm{T}}^{-1} \sum\nolimits_{i=1}^n N_i^2 + (C_\sigma C_{K'})^2 (c_\sigma c_K)^{-2}.$$

Since for any fixed n, $\mathsf{E}(N_{\mathrm{T}}^{-1} \sum_{i=1}^n N_i^2) < \infty$, the Dominated Convergence Theorem shows that $r_{1,1}(t,t) = \mathsf{E}\, R_{1,1}(t,t)$. Further, one has $I_1(t) = h\,\mathsf{E}\, N_1 \sigma_Y^2(th) \int K'(v)^2\, dv + U(h^3) + U_{a.s.}\{\sqrt{\log n/(nh)}\}$, $I_2(t) = h\,\mathsf{E}\, N_1 \sigma_Y^2(th) \int K(v)^2\, dv + U(h^3) + U_{a.s.}\{\sqrt{\log n/(nh)}\}$, $I_3(t) = U(h^4) + U_{a.s.}\{\sqrt{\log n/(nh)}\}$ which by (37) imply

$$\sup_{t \in [1, h^{-1}-1]} |R(t,t)_{11} - C(K)| = \mathcal{O}(h^2) + \mathcal{O}_{a.s.}\{\sqrt{\log n/(nh)}\},$$

note that $C(t) = C(K)$ given in (3.5) for $t \in \left[1, h^{-1} - 1\right]$. Clearly,

$$\sup_{t \in [1, h^{-1}-1]} |r_{1,1}(t,t) - C(K)| \leq \mathsf{E} \sup_{t \in [1, h^{-1}-1]} |R(t,t)_{11} - C(K)|,$$

and hence, in order to show $\lim_{n\to\infty} \sup_{t\in[1,h^{-1}-1]} |r_{1,1}(t,t) - C(K)| = 0$, it is sufficient to show uniform integrability of $\sup_{t\in[1,h^{-1}-1]} |R(t,t)_{11} - C(K)|$. In fact, since $\sup_{t\in[1,h^{-1}-1]} |R(t,t)_{11}| = \mathcal{O}(N_{\mathrm{T}}^{-1} \sum_{i=1}^n N_i^2)$, then $\exists$ constant C such that

$$\lim_{M\to\infty} \sup_n \mathsf{E} \left\{ \sup\nolimits_{t\in[1,h^{-1}-1]} |R(t,t)_{11} - C(K)|\, I \left(\sup\nolimits_{t\in[1,h^{-1}-1]} |R(t,t)_{11} - C(K)| > M \right) \right\}$$
$$\leq C \lim_{M'\to\infty} \sup_n \mathsf{E} \left\{ \left| N_{\mathrm{T}}^{-1} \sum\nolimits_{i=1}^n N_i^2 \right| I \left(\left| N_{\mathrm{T}}^{-1} \sum\nolimits_{i=1}^n N_i^2 \right| \right) > M' \right\} \leq$$
$$C \lim_{M'\to\infty} \sup_n \mathsf{E} \left(n^{-1} \sum\nolimits_{i=1}^n N_i^2 \right)^2 M'^{-1} \leq C \lim_{M'\to\infty} \sup_n \frac{n \sum_{i=1}^n \mathsf{E}\, N_i^4}{n^2} M'^{-1} = 0.$$

This completes the proof of (30).

To prove (31), similar to (30), one first shows that

$$\lim_{n\to\infty}\sup_{t,s\in[1,h^{-1}-1]}\left|r(t,s)-\frac{K*K(t-s)}{\|K\|_2\left\{\int K^2(x+t-s)\,dx\right\}^{1/2}}\right|=0. \tag{38}$$

To investigate $\inf_{t,s\in[1,h^{-1}-1],|t-s|<2} r(t,s)$, one notes the simple limit

$$\lim_{|t-s|\to 2-}\frac{K*K(t-s)}{\|K\|_2\left\{\int K^2(x+t-s)\,dx\right\}^{1/2}}=0, \tag{39}$$

which implies that for $\forall\varepsilon\in(0,1)$, $\exists\ \delta\in(0,1)$ such that

$$\sup_{t,s\in[1,h^{-1}-1],2-\delta\le|t-s|<2}\left|\frac{K*K(t-s)}{\|K\|_2\left\{\int K^2(x+t-s)\,dx\right\}^{1/2}}\right|<\frac{\varepsilon}{2}. \tag{40}$$

Note next that if $|t-s|\le 2-\delta, t,s\in\left[1,h^{-1}-1\right]$ then the set $\{x: K(x+t-s)>0\}$ has Lebesgue measure $\ge\delta>0$ and $0<K*K(t-s)\le\left\{\int K^2(x)\,dx\right\}^{1/2}\left\{\int K^2(x+t-s)\,dx\right\}^{1/2}$ (Cauchy–Schwarz inequality). Therefore

$$\inf_{t,s\in[1,h^{-1}-1],|t-s|\le 2-\delta}\frac{K*K(t-s)}{\|K\|_2\left\{\int K^2(x+t-s)\,dx\right\}^{1/2}}\ge 0. \tag{41}$$

Now (40) and (41) imply that

$$\inf_{t,s\in[1,h^{-1}-1],|t-s|<2}\frac{K*K(t-s)}{\|K\|_2\left\{\int K^2(x+t-s)\,dx\right\}^{1/2}}>-\frac{\varepsilon}{2}, \tag{42}$$

while (38) implies that for large n

$$\sup_{t,s\in[1,h^{-1}-1]}\left|r(t,s)-\frac{K*K(t-s)}{\|K\|_2\left\{\int K^2(x+t-s)\,dx\right\}^{1/2}}\right|\le\frac{\varepsilon}{2}. \tag{43}$$

Therefore (42) and (43) entail that for $c=1-\varepsilon\in(0,1)$

$$\inf_{t,s\in[1,h^{-1}-1],|t-s|<2} r(t,s)\ge-\frac{\varepsilon}{2}-\frac{\varepsilon}{2}=-\varepsilon=-1+c>-1. \tag{44}$$

Now we investigate $\sup_{2>|t-s|\ge\delta,\ t,s\in[1,h^{-1}-1]} r(t,s)$. As claimed for (41), if $|t-s|\le 2-\delta$, $0<K*K(t-s)\le\left\{\int K^2(x)\,dx\right\}^{1/2}\left\{\int K^2(x+t-s)\,dx\right\}^{1/2}$. Further, if $|t-s|\ge\delta$, the inequality becomes strict since the two continuous functions $K(x)$ and $K(x+t-s)$ are not proportional. Since the continuous function of $t-s$ satisfies

$$\frac{K*K(t-s)}{\|K\|_2\left\{\int K^2(x+t-s)\,dx\right\}^{1/2}}<1, 2-\delta\ge|t-s|\ge\delta$$

hence $\exists\ c' > 0$ such that

$$\sup_{2-\delta\geq|t-s|\geq\delta,\ t,s\in[1,h^{-1}-1]} \frac{K*K\left(t-s\right)}{\|K\|_2\left\{\int K^2\left(x+t-s\right)dx\right\}^{1/2}} \leq 1-c' < 1. \tag{45}$$

In fact, (40) and (45) show that

$$\sup_{2>|t-s|\geq\delta,\ t,s\in[1,h^{-1}-1]} \frac{K*K\left(t-s\right)}{\|K\|_2\left\{\int K^2\left(x+t-s\right)dx\right\}^{1/2}} \leq \max\left(1-c',\varepsilon/2\right) < 1. \tag{46}$$

Based on (38) and (46), for large n, $\exists$ a constant $c\in(0,1)$ such that

$$\sup_{2>|t-s|\geq\delta,\ t,s\in[1,h^{-1}-1]} r\left(t,s\right) < 1-c. \tag{47}$$

(44) and (47) complete the proof of (31).

To prove (32), let $t,s\in\left[1,h^{-1}-1\right]$, we first investigate $\sup_{0<|t-s|<\delta} r_{1,0}\left(t,s\right)/\left(t-s\right)$. Taylor's Theorem shows that $\exists$ constants $L_K, L'_K, l_K, l'_K, C_1$ such that

$$\left(1-\left[K*K\left(t-s\right)\|K\|_2^{-1}\left\{\int K^2\left(x+t-s\right)dx\right\}^{-1/2}\right]^2\right)\left(t-s\right)^{-2} \leq \tag{48}$$

$$\left[1-\left\{\|K\|_2+\int K\left(x\right)K''\left(x\right)dx\left(t-s\right)^2/2+L_K\delta^3\right\}^2\left\{\|K\|_2^2+\|K\|_2^2\int K^2\left(x\right)''dx\left(t-s\right)^2/2+L'_K\delta^3\right\}^{-1}\right]\left(t-s\right)^{-2}$$

$$\leq\left[1-\|K\|_2+l_K\left(t-s\right)^2\left\{\|K\|_2^2+l'_K\left(t-s\right)^2\right\}^{-1}\right]\left(t-s\right)^{-2}\leq\left(l'_K-l_K\right)\left\{\|K\|_2^2+l'_K\left(t-s\right)^2\right\}^{-1}\leq C_1.$$

Clearly, one can make $l'_K-l_K>0$ and therefore

$$\sup_{0<|t-s|<\delta}\left(1-\left[K*K\left(t-s\right)\|K\|_2^{-1}\left\{\int K^2\left(x+t-s\right)dx\right\}^{-1/2}\right]^2\right)\left(t-s\right)^{-2}\leq C_1. \tag{49}$$

Based on (38) and (49), similar to the proof of (44), for large n

$$\sup_{0<|t-s|<\delta}\frac{1-r^2\left(t,s\right)}{\left(t-s\right)^2}\leq C_1. \tag{50}$$

Next, we consider $\sup_{0<|t-s|<\delta}\ \frac{r_{1,0}(t,s)}{t-s}$, similar to (38), one has

$$\lim_{n\to\infty}\sup_{t,s\in[1,h^{-1}-1]}\left|r_{1,0}\left(t,s\right)-\frac{-K'*K\left(t-s\right)}{\|K\|_2\left\{\int K^2\left(x+t-s\right)dx\right\}^{1/2}}\right|=0. \tag{51}$$

Based on (51), similar to the proof of (50), for large n , $\exists C_2>0$ such that

$$\sup_{0<|t-s|<\delta}\frac{r_{1,0}\left(t,s\right)}{t-s}\leq C_2. \tag{52}$$

Clearly, (50) and (52) implies for $C = \max\{C_1, C_2\}$

$$\sup_{0<|t-s|<\delta} \max\left\{\frac{r_{1,0}(t,s)}{t-s}, \frac{1-r^2(t,s)}{(t-s)^2}\right\} \le C. \tag{53}$$

Finally, $\inf_{0<|t-s|<\delta, t,s\in[0,h^{-1}]} \min\left\{\frac{r_{1,0}(t,s)}{t-s}, \frac{1-r^2(t,s)}{(t-s)^2}\right\} \ge c$ and (33) can be similarly proved.

To prove (34), similar to (38), we make use of

$$\lim_{n\to\infty} \sup_{t,s\in[1,h^{-1}-1]} \left| \frac{r_{1,0}^2(t,s)}{1-r^2(t,s)} - \frac{\{K' * K(t-s)\}^2}{\|K\|_2^2 \left\{\int K^2(x+t-s)\,dx\right\} - \{K * K(t-s)\}^2} \right| = 0. \tag{54}$$

Similar to (39), one uses the simple limit

$$\lim_{|t-s|\to 2^-} \left| \frac{\{K' * K(t-s)\}^2}{\|K\|_2^2 \left\{\int K^2(x+t-s)\,dx\right\} - \{K * K(t-s)\}^2} \right| = 0,$$

which implies that for constant C_1 and $\forall \varepsilon \in (0, C_1)$, $\exists\ \delta > 0$ such that

$$\sup_{2-\delta\le|t-s|<2} \left| \frac{\{K' * K(t-s)\}^2}{\|K\|_2^2 \left\{\int K^2(x+t-s)\,dx\right\} - \{K * K(t-s)\}^2} \right| < \varepsilon < C_1. \tag{55}$$

For $2-\delta > |t-s| \ge \delta > 0$, Cauchy–Schwarz inequality provides $C_2 > 0$ such that

$$\sup_{t,s\in[1,h^{-1}-1],\ 2-\delta>|t-s|\ge\delta>0} \{K' * K(t-s)\}^2 \le \|K'\|_2^2 \|K\|_2^2 \le C_2.$$

By (45), $\exists\ C_3 > 0$ such that

$$\inf_{t,s\in[1,h^{-1}-1],\ 2-\delta>|t-s|\ge\delta>0} \|K\|_2^2 \left\{\int K^2(x+t-s)\,dx\right\} - \{K * K(t-s)\}^2 \ge C_3 > 0.$$

Therefore, one obtains the following bound on the ratio

$$\sup_{t,s\in[1,h^{-1}-1],\ 2-\delta>|t-s|\ge\delta>0} \frac{\{K' * K(t-s)\}^2}{\|K\|_2^2 \left\{\int K^2(x+t-s)\,dx\right\} - \{K * K(t-s)\}^2} \le \frac{C_2}{C_3}. \tag{56}$$

For $|t-s| < \delta$, Taylor expansion shows that $\exists\ C_4, L_K, l_K > 0$

$$\begin{aligned} &\sup_{t,s\in[1,h^{-1}-1],\ |t-s|<\delta} \frac{\{K' * K(t-s)\}^2}{\|K\|_2^2 \left\{\int K^2(x+t-s)\,dx\right\} - \{K * K(t-s)\}^2} \\ &\le \frac{\|K'\|_2^4 (t-s)^2 + L_K \delta^3}{\|K\|_2^2 \|K'\|_2^2 (t-s)^2 + l_K \delta^3} \le C_4. \end{aligned} \tag{57}$$

Finally, (55)-(57) show that $\exists\, C > 0$ such that

$$\sup_{t,s\in[1,h^{-1}-1],\ 2>|t-s|>\delta}\left|\frac{\{K'*K(t-s)\}^2}{\|K\|_2^2\left\{\int K^2(x+t-s)\,dx\right\}-\{K*K(t-s)\}^2}\right|\leq C. \tag{58}$$

As a result, similar to prove (50), (54) and (58) imply (34).

The proof of (35) is similar to (34).

8. Proof of Lemma A.14

Lemma A.14 Under Assumptions (A1)-(A6), let $\Delta_{3,n}(x)=\widetilde{\sigma}_n(x)\,\sigma_n^{-1}(x)-1, x\in[0,1]$, then $\Delta_{3,n}(x)=\mathcal{U}(h)+\mathcal{U}_{a.s.}\{\sqrt{\log n/(nh^2)}\}$ and for $\widehat{\varepsilon}(x)$, $\sigma_n^2(x)$ given in (3.4) as $n\to\infty$

$$\sup_{[0,1]}\left|\sigma_n^{-1}(x)\,\{\widehat{\varepsilon}_{N_{\mathrm{T}}}(x)+\widehat{\xi}(x)\}-\eta(x)\right|=\sup_{[0,1]}|\Delta_{3,n}(x)|\,|\eta(x)|=\mathcal{O}_p\{h\sqrt{\log n}+\sqrt{\log^2 n/(nh^2)}\}.$$

Proof. It follows from the definition of $\eta(x)$ in (A.9) that $|\Delta_{3,n}(x)|=\left|\widetilde{\sigma}_n(x)\,\sigma_n^{-1}(x)-1\right|\leq\left|\widetilde{\sigma}_n^2(x)\,\sigma_n^{-2}(x)-1\right|$ in which

$$\widetilde{\sigma}_n^2(x)=f^{-2}(x)\,D_x^{-2}(K)\,N_{\mathrm{T}}^{-1}\{N_{\mathrm{T}}^{-1}\sum\nolimits_{i,j}R_{ij,\varepsilon}^2(x)+N_{\mathrm{T}}^{-1}\sum\nolimits_{i}R_{i,\xi}^2(x)\}.$$

By Lemma A.10, one shows that $N_{\mathrm{T}}^{-1}\sum_{i,j}R_{ij,\varepsilon}^2(x)=\mathsf{E}\,R_{ij,\varepsilon}^2(x)+\mathcal{U}_{a.s}\{\sqrt{\log n/(nh^2)}\}$ and $N_{\mathrm{T}}^{-1}\sum_i R_{i,\xi}^2(x)=(\mathsf{E}\,N_1)^{-1}\,\mathsf{E}\,R_{i,\xi}^2(x)+\mathcal{U}_{a.s.}\{\sqrt{\log n/(nh^2)}\}$. Also, $N_{\mathrm{T}}=n\,\mathsf{E}\,N_1+\mathcal{U}_{a.s.}(\sqrt{\log n/n})$ and $\mathsf{E}\,R_{ij,\varepsilon}^2(x)+(\mathsf{E}\,N_1)^{-1}\,\mathsf{E}\,R_{i,\xi}^2(x)=\mathcal{U}\left(h^{-1}\right)$. Therefore,

$$\begin{aligned}\widetilde{\sigma}_n^2(x)&=f^{-2}(x)\,D_x^{-2}(K)\,[(n\,\mathsf{E}\,N_1)^{-1}+\mathcal{U}_{a.s.}(\sqrt{\log n/n})]\times\\&[\mathsf{E}\,R_{ij,\varepsilon}^2(x)+(\mathsf{E}\,N_1)^{-1}\,\mathsf{E}\,R_{i,\xi}^2(x)+\mathcal{U}_{a.s.}\{\sqrt{\log n/(nh^2)}\}]\\&=f^{-2}(x)\,D_x^{-2}(K)\,(n\,\mathsf{E}\,N_1)^{-1}\,\{\mathsf{E}\,R_{ij,\varepsilon}^2(x)+(\mathsf{E}\,N_1)^{-1}\,\mathsf{E}\,R_{i,\xi}^2(x)\}+\\&\mathcal{U}_{a.s.}(\sqrt{\log n/n})\{\mathsf{E}\,R_{ij,\varepsilon}^2(x)+(\mathsf{E}\,N_1)^{-1}\,\mathsf{E}\,R_{i,\xi}^2(x)\}=\sigma_n^2(x)+\mathcal{U}_{a.s.}\{\sqrt{\log n/(nh^2)}\},\end{aligned}$$

which implies that $\widetilde{\sigma}_n^2(x)\,\sigma_n^{-2}(x)=1+\mathcal{U}_{a.s.}\{\sqrt{\log n/(nh^2)}\}$ and hence this lemma holds.