SIMULTANEOUS CONFIDENCE BANDS FOR MEAN AND VARIANCE FUNCTIONS BASED ON DETERMINISTIC DESIGN

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Supplementary Materials

This supplement contains all the proofs for the main results and some figures for the simulated and real data examples.

S1 Preliminaries

In this section, we use c and C to denote any positive constants in the generic sense. We will need the following lemmas in the proofs of our main results.

Lemma S.1. *(Komlós, Major, and Tusnády, 1976, Theorem 4) Suppose* $\xi_i, 1 \leq i < \infty$, *are i.i.d. r.v.'s with* $\mathrm{E}\,\xi_1 = 0, \mathrm{E}\,\xi_1^2 = 1$. *Let* $H(x) > 0, x > 0$ *be a monotone increasing and continuous function such that for constants* $\delta > 0, x_0 > 0$, $x^{-3-\delta}H(x)$ *is monotone increasing for* $x > x_0$, *and* $x^{-1}\log H(x)$ *is monotone decreasing for* $x > x_0$. *Define* K_n *by the equation*

$H(K_n) = n$. If $\mathrm{E}\, H(|\xi_1|) < \infty$, then there exist constants $C_1, C_2, a > 0$ depending only on the distribution of ξ_1 and a sequence $\{Z_i\}_{i=1}^n$ of i.i.d. r.v.'s with standard normal distribution such that for any t, $t > K_n, t^2/\log H(t) < C_1 n$,

$$P\left\{\max_{1\le l\le n} |S_l - W_l| > t\right\} \le C_2 n \left\{H(at)\right\}^{-1},$$

where $S_l = \sum_{i=1}^l \xi_i$ and $W_l = \sum_{i=1}^l Z_i$.

Lemma S.2. *Assumption (M3) holds under Assumption (M3').*

Proof of Lemma S.2 Under Assumption (M3') that $\mathrm{E}\,|\varepsilon_1|^{2+\eta} < +\infty$ for some $\eta > 2/\beta - 2, \beta \in (0, 1/2 - 1/(4\theta + 4p - 2))$, hence $\eta > 2$. Now let $H(x) = x^{2+\eta}$ which is increasing for $x > 0$. If one sets $x_0 = 3$, then for $x \ge x_0, 0 < \delta < 1$, $x^{-3-\delta}H(x)$ is monotone increasing, $x^{-1}\log H(x)$ is monotone decreasing, hence all conditions of Lemma S.1 are met. Let $K_n = n^{1/(2+\eta)}, \gamma = (\eta+2)\beta - 1 > 1, t = t_n = n^\beta$, then Lemma S.1 entails that there exist constants $C_1, C_2, a > 0$ depending only on the distribution of ε_1 and the standard normal variables $\{Z_{in}\}_{i=1}^n$ such that for $t_n > K_n, t_n^2/\log H(t_n) < C_1 n$, $n/H(at_n) = (aC_1)^{-2-\eta} n^{1-(2+\eta)\beta} = a^{-2-\eta}n^{-\gamma}$, let $C_0 = C_2(C_1 a)^{-2-\eta}$, then

$$P\left\{\max_{1\le l\le n} |S_l - W_{l,n}| > n^\beta\right\} \le C_0 n^{-\gamma}, S_l = \sum_{i=1}^l \varepsilon_i, W_{l,n} = \sum_{i=1}^l Z_{in}, \quad \text{(S1.1)}$$

which implies Lemma S.2.

In the next following lemmas, we use $\mathcal{O}(1)$ to mean 'bounded for any fixed $x \in [0,1]$', $\mathcal{O}_{a.s.}(1)$ $[o_{a.s.}(1)]$ to mean 'bounded [tends to 0] almost surely for any fixed $x \in [0,1]$' and $U_p(1)$ $[u_p(1)]$ to mean 'bounded [tends to 0] in probability uniformly for any $x \in [0,1]$'.

Lemma S.3. *Under Assumption (M3), as $n \to \infty$, $S_n, W_{n,n}$ in Equation (S1.1) satisfy*

$$\left|n^{-1}S_n - n^{-1}W_{n,n}\right| = \mathcal{O}_{a.s.}\left(n^{\beta-1}\right).$$

Proof of Lemma S.3 See the proof of Lemma A.5 in Cao et al. (2012).

For any Lebesgue measurable function $\phi(x)$ on $[0,1]$, denote $\|\phi(x)\|_\infty = \sup_{x\in[0,1]} |\phi(x)|$ and denote a class of Lipschitz continuous functions by $\mathrm{Lip}([0,1], C) = \{\phi(x) | |\phi(x) - \phi(x')| \le C|x-x'|, \forall x, x' \in [0,1], C > 0\}$.

Lemma S.4. *Under Assumption (E1), there exists a constant $C_p > 0$ with $p > 1$, such that for any $m \in C^p[0,1]$, there exists a function $g \in \mathcal{H}_N^{(p-2)}$ for which $\|g - m\|_\infty \le C_p N^{-p}$ and $g - m \in \mathrm{Lip}([0,1], C_p N^{1-p})$. Furthermore, the function $\tilde{m}_p(x)$ given in (3.8) satisfies:*

$$\|\tilde{m}_p(x) - m(x)\|_\infty \le C \inf_{g\in\mathcal{H}_N^{(p-2)}} \|g(x) - m(x)\|_\infty = \mathcal{O}\left(N^{-p}\right).$$

Proof of Lemma S.4 See de Boor (2001) Theorem 6 on p.149 and Theorem 26 on p.155 for the detailed proofs.

The following result is based on Lemma A.3 of Song and Yang (2009) and Lemma A.1 of Xue and Yang (2006) in which $\|\boldsymbol{\varsigma}\|$ represents the Euclidean norm for any vector $\boldsymbol{\varsigma}$.

Lemma S.5. *Under Assumption (E7), there exists constants c and C independent of n such that for any vector $\eta = \left\{\eta_{1-p}, \ldots, \eta_N\right\}^T \in \mathbb{R}^{N+p}$,*

$$c\sum_{j=1-p}^{N} \eta_j^2 \le \left\|\sum_{j=1-p}^{N} \eta_j B_{j.p}(x)\right\|_2^2 \le C\sum_{j=1-p}^{N} \eta_j^2, \tag{S1.2}$$

and for large n,

$$c\|\eta\|^2 \le \eta^T \left(n^{-1}\mathbf{B}^T\mathbf{B}\right)^{-1} \eta \le C\|\eta\|^2. \tag{S1.3}$$

Lemma S.6. *Under Assumptions (E2) and (E7), as $n \to \infty$, the function $\tilde{\varepsilon}_p(x)$ given in (3.7) satisfies $\|\tilde{\varepsilon}_p(x)\|_\infty = \mathcal{O}_p\left(n^{-1/2}N^{1/2}\right)$.*

Proof of Lemma S.6 According to Equation (3.9), $\tilde{\varepsilon}_p(x)$ can be written as

$$\{B_{J,p}(x)\}_{J=1-p}^{N}\left(n^{-1}\mathbf{B}^{T}\mathbf{B}\right)^{-1}\left\{n^{-1}\sum\nolimits_{i=1}^{n}B_{J,p}\left(i/n\right)\sigma\left(i/n\right)\varepsilon_{i}\right\}_{J=1-p}^{N}.$$

By applying Equation (S1.2) in Lemma S.5, $\tilde{\varepsilon}_p(x)$ is bounded by

$$C\left\|\{B_{J,p}(x)\}_{j=1-p}^{N}\right\|\left\|\left\{n^{-1}\sum\nolimits_{i=1}^{n}B_{J,p}\left(i/n\right)\sigma\left(i/n\right)\varepsilon_{i}\right\}_{J=1-p}^{N}\right\|.$$

The definition of $\{B_{J,p}(x)\}_{j=1-p}^{N}$ and the simple fact that $\|b_{J,p}(x)\|_2 \geq cN^{-1/2}$, $1-p \leq J \leq N$, for some constant c imply that $\sup_{x\in[0,1]}\left\|\{B_{J,p}(x)\}_{J=1-p}^{N}\right\| = \mathcal{O}\left(N^{1/2}\right)$. Hence,

$$\sup_{x\in[0,1]}\left|\tilde{\varepsilon}_p\left(x\right)\right| = O\left(N^{1/2}\right)\left\|\left\{n^{-1}\sum\nolimits_{i=1}^{n}B_{J,p}\left(i/n\right)\sigma\left(i/n\right)\varepsilon_{i}\right\}_{J=1-p}^{N}\right\|. \quad \text{(S1.4)}$$

Let $[x]$ represent the integer part of x. Notice that

$$\begin{aligned}
&\mathrm{E}\left\|\left\{n^{-1}\sum\nolimits_{i=1}^{n}B_{J,p}\left(i/n\right)\sigma\left(i/n\right)\varepsilon_{i}\right\}_{J=1-p}^{N}\right\|^{2}\\
&= \mathrm{E}\sum\nolimits_{J=1-p}^{N}\left\{n^{-1}\sum\nolimits_{i=1}^{n}B_{J,p}\left(i/n\right)\sigma\left(i/n\right)\varepsilon_{i}\right\}^{2}\\
&= \sum\nolimits_{J=1-p}^{N}n^{-2}\sum\nolimits_{i=1}^{n}B_{J,p}^{2}\left(i/n\right)\sigma^{2}\left(i/n\right)\\
&\leq C_{\sigma}^{2}\left(N+p\right)n^{-2}C\left(\left[pnN^{-1}\right]+1\right) = \mathcal{O}\left(n^{-1}\right).
\end{aligned}$$

Thus, $\left\|\{n^{-1}\sum_{i=1}^{n}B_{J,p}\left(i/n\right)\sigma\left(i/n\right)\varepsilon_{i}\}_{J=1-p}^{N}\right\| = \mathcal{O}_p\left(n^{-1/2}\right)$ which together with (S1.4) implies that

$$\sup_{x\in[0,1]}\left|\tilde{\varepsilon}_p\left(x\right)\right| = \mathcal{O}_p\left(n^{-1/2}N^{1/2}\right).$$

Lemma S.7. *Under Assumptions (E2), (E4) and (E6), we denote* $\|\varphi_n\|_\infty = \rho_n$ *for any function* φ_n *with domain* $[0,1]$*. Then for any i.i.d.* $N\left(0,1\right)$ *variables* $\{Z_{in}\}_{i=1}^{n}$ *satisfying (E6), as* $n\to\infty$*,*

$$\sup_{x\in[0,1]}\left|n^{-1}\sum\nolimits_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n-x\right)\sigma\left(i/n\right)\varphi_n\left(i/n\right)Z_{in}\right| = \mathcal{O}_p\left(n^{-1/2}\tilde{h}^{-1/2}\rho_n\log^{1/2}n\right).$$

Proof of Lemma S.7 Let $\Lambda_n(x) = n^{-1} \sum_{i=1}^{n} \tilde{K}_{\tilde{h}}(i/n - x)\, \sigma(i/n)\, \varphi_n(i/n)\, Z_{in}$. Obviously, $\Lambda_n(x)$ is the Gaussian process with mean 0 and variance

$$\begin{aligned}
\mathrm{E}\left\{\Lambda_n(x)\right\}^2 &= n^{-2} \sum\nolimits_{i=1}^{n} \tilde{K}_{\tilde{h}}^2(i/n - x)\, \sigma^2(i/n)\, \varphi_n^2(i/n) \\
&\leq C_\sigma^2 \rho_n^2 n^{-2} \sum\nolimits_{i=1}^{n} \tilde{K}_{\tilde{h}}^2(i/n - x) \\
&\leq C_\sigma^2 \rho_n^2 n^{-2} \tilde{h}^{-2} \left\|\tilde{K}\right\|_\infty^2 \left(\left[2n\tilde{h}\right] + 1\right) \\
&\leq C_\sigma^2 \rho_n^2 \left\|\tilde{K}\right\|_\infty n^{-1} \tilde{h}^{-1}.
\end{aligned}$$

In the following, we use the well-known tail property of the normal distribution, i.e., $1 - \Phi(x) \leq \phi(x)/x$ for $x \geq 0$, in which $\Phi(x)$ and $\phi(x)$ are the cumulative distribution function and the density function of the standard normal respectively. Hence there exists some $c > 0$ such that $1 - \Phi(x) \leq c\phi(x)$ for large x. Take $x = \delta_n = \sqrt{16 \log n}$, and hence there exists a constant c such that for a large enough n,

$$P\left\{|\Lambda_n(x)| \geq C\rho_n n^{-1/2} \tilde{h}^{-1/2} \delta_n\right\} \leq c \exp\left\{-\delta_n^2/2\right\} = cn^{-8}.$$

Divide the interval $[0,1]$ into $M_n = n^4$ equally spaced intervals with disjoint endpoints $0 = x_0 < x_1 < \cdots < x_{M_n} = 1$, so that the consecutive endpoints make a total of M_n subintervals with length M_n^{-1}. One immediately obtains that

$$\begin{aligned}
&P\left\{\max_{0 \leq k \leq M_n} |\Lambda_n(x_k)| \geq C\rho_n n^{-1/2} \tilde{h}^{-1/2} \delta_n\right\} \\
&\qquad \leq \sum_{k=0}^{M_n} P\left\{|\Lambda_n(x_k)| \geq C\rho_n n^{-1/2} \tilde{h}^{-1/2} \delta_n\right\} \\
&\qquad \leq \sum_{k=0}^{M_n} cn^{-8} = cn^{-4}.
\end{aligned}$$

Thus, $\sum_{n=1}^{\infty} P\left\{\max_{0 \leq k \leq M_n} |\Lambda_n(x_k)| \geq C\rho_n n^{-1/2} \tilde{h}^{-1/2} \delta_n\right\} < +\infty$. The Borrel-

Cantelli Lemma implies

$$\max_{0\le k\le M_n} \left|\Lambda_n\left(x_k\right)\right| = \mathcal{O}_{a.s}\left(n^{-1/2}\tilde{h}^{-1/2}\rho_n \log^{1/2} n\right). \tag{S1.5}$$

Taking the supremum over the whole interval $x\in[0,1]$, one obtains that

$$\max_{x\in[0,1]} \left|\Lambda_n(x)\right| \le \max_{0\le k\le M_n}\left|\Lambda_n\left(x\right)-\Lambda_n\left(x_k\right)\right| + \max_{0\le k\le M_n-1}\sup_{x\in[x_k,x_{k+1}]}\left|\Lambda_n\left(x_k\right)\right|. \tag{S1.6}$$

Meanwhile, $\max_{0\le k\le M_n-1}\sup_{x\in[x_k,x_{k+1}]}\left|\Lambda_n(x)-\Lambda_n\left(x_k\right)\right|$ is bounded by

$$\begin{aligned}&\max_{0\le k\le M_n-1}\sup_{x\in[x_k,x_{k+1}]} n^{-1}\sum_{i=1}^n\left|\tilde{K}_{\tilde{h}}\left(x-i/n\right)-\tilde{K}_{\tilde{h}}\left(x_k-i/n\right)\right|\rho_n\sigma\left(i/n\right)\left|Z_{in}\right|\\ &\le C_\sigma n^{-1}\tilde{h}^{-2}\left\|\tilde{K}^{(1)}\right\|_\infty \rho_n M_n^{-1}\sum_{i=1}^n\left|Z_{in}\right| = o_p\left(n^{-2}\rho_n\right).\end{aligned} \tag{S1.7}$$

Consequently, Equations (S1.5), (S1.6) and (S1.7) establish Lemma **S.7.**

Lemma S.8. *Under Assumptions (E2), (E4) and (E6), for any function* $\varphi_n\in\mathrm{Lip}\left([0,1],C_{\varphi,n}\right)$ *and* $\|\varphi_n\|_\infty=\rho_n$, *as* $n\to\infty$,

$$\begin{aligned}&\sup_{x\in[0,1]}\left|n^{-1}\sum\nolimits_{i=1}^n\tilde{K}_{\tilde{h}}\left(i/n-x\right)\sigma\left(i/n\right)\varphi_n\left(i/n\right)\varepsilon_i\right|\\ &= \mathcal{O}_p\left(n^{-1/2}\tilde{h}^{-1/2}\rho_n\log^{1/2}n + n^{\beta-1}C_{\varphi,n} + n^{\beta-1}\tilde{h}^{-1}\rho_n\right).\end{aligned}$$

Proof of Lemma S.8 According to Equation (S1.1), it is obvious that

$$\varepsilon_i = S_i - S_{i-1}, Z_{in} = W_{i,n} - W_{i-1,n}, 1\le i\le n,$$

where $S_0=0, W_{0,n}=0$. Denote $\Upsilon_n\left(i,x\right)=\tilde{K}_{\tilde{h}}\left(i/n-x\right)\sigma\left(i/n\right)\varphi_n\left(i/n\right)$, hence

$$n^{-1}\sum\nolimits_{i=1}^n\tilde{K}_{\tilde{h}}\left(i/n-x\right)\sigma\left(i/n\right)\varphi_n\left(i/n\right)\varepsilon_i = n^{-1}\sum\nolimits_{i=1}^n\Upsilon_n\left(i,x\right)\left(S_i-S_{i-1}\right)$$

and

$$n^{-1}\sum\nolimits_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n-x\right)\sigma\left(i/n\right)\varphi_{n}\left(i/n\right)Z_{in}$$
$$=n^{-1}\sum\nolimits_{i=1}^{n}\Upsilon_{n}\left(i,x\right)\left(W_{i,n}-W_{i-1,n}\right).$$

Thus, $n^{-1}\sum_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n-x\right)\sigma\left(i/n\right)\varphi_{n}\left(i/n\right)\varepsilon_{i}-n^{-1}\sum_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n-x\right)\sigma\left(i/n\right)$ $\times\varphi_{n}\left(i/n\right)Z_{in}$ is bounded by

$$\sup_{x\in[0,1]}\left|n^{-1}\sum\nolimits_{i=1}^{n}\Upsilon_{n}\left(i,x\right)\varepsilon_{i}-n^{-1}\sum\nolimits_{i=1}^{n}\Upsilon_{n}\left(i,x\right)Z_{in}\right|$$
$$\leq\sup_{x\in[0,1]}\left|\Gamma_{n}(x)\right|+\sup_{x\in[0,1]}\left|n^{-1}\Upsilon_{n}\left(n,x\right)\left(S_{n}-W_{n,n}\right)\right|,\qquad\text{(S1.8)}$$

where $\Gamma_{n}(x)=n^{-1}\sum_{i=1}^{n-1}\left\{\Upsilon_{n}\left(i,x\right)-\Upsilon_{n}\left(i+1,x\right)\right\}\left(S_{i}-W_{i,n}\right)$. Assumption (E2) and $\varphi_{n}(x)\in\mathrm{Lip}\{[0,1],C_{\varphi,n}\}$ lead to

$$\sup_{1\leq i\leq n-1}\left|\sigma\left(i/n\right)\varphi_{n}\left(i/n\right)-\sigma\left(\left(i+1\right)/n\right)\varphi_{n}\left(\left(i+1\right)/n\right)\right|=\mathcal{O}\left(n^{-1}C_{\varphi,n}+\rho_{n}n^{-1}\right).$$

Therefore,

$$\sup_{1\leq i\leq n-1}\left|\Upsilon_{n}\left(i,x\right)-\Upsilon_{n}\left(i+1,x\right)\right|$$
$$=\sup_{1\leq i\leq n-1}\left|\tilde{K}_{\tilde{h}}\left(i/n-x\right)\sigma\left(i/n\right)\varphi_{n}\left(i/n\right)\right.$$
$$\left.-\tilde{K}_{\tilde{h}}\left(\left(i+1\right)/n-x\right)\sigma\left(\left(i+1\right)/n\right)\varphi_{n}\left(\left(i+1\right)/n\right)\right|$$
$$\leq\sup_{1\leq i\leq n-1}\left|\left\{\tilde{K}_{\tilde{h}}\left(i/n-x\right)-\tilde{K}_{\tilde{h}}\left(\left(i+1\right)/n-x\right)\right\}\sigma\left(i/n\right)\varphi_{n}\left(i/n\right)\right.$$
$$\left.+U\left(n^{-1}\tilde{h}^{-1}C_{\varphi,n}+\rho_{n}\tilde{h}^{-1}n^{-1}\right)\right|$$
$$=U\left(\tilde{h}^{-2}n^{-1}\rho_{n}+n^{-1}\tilde{h}^{-1}C_{\varphi,n}\right).\qquad\text{(S1.9)}$$

According to Assumption (E6) and Equation (S1.9), for the first term of Equation (S1.8), one obtains that

$$\sup_{x\in[0,1]}\left|\Gamma_{n}\left(x\right)\right|\leq n^{-1}\left(\left[2n\tilde{h}\right]+1\right)\sup\nolimits_{1\leq i\leq n-1}\left|\Upsilon_{n}\left(i,x\right)-\Upsilon_{n}\left(i+1,x\right)\right|$$
$$\times\max\nolimits_{1\leq i\leq n-1}\left|S_{i}-W_{i,n}\right|=\mathcal{O}_{p}\left(\tilde{h}^{-1}n^{\beta-1}\rho_{n}+n^{\beta-1}C_{\varphi,n}\right).\qquad\text{(S1.10)}$$

For the second term of Equation (S1.8), according to Lemma S.3,

$$\sup_{x\in[0,1]}\left|n^{-1}\Upsilon_n\left(1,x\right)\left(S_n-W_{n,n}\right)\right|=\mathcal{O}_p\left(n^{\beta-1}\tilde{h}^{-1}\rho_n\right). \tag{S1.11}$$

Consequently, Equations (S1.8), (S1.10), (S1.11) and Lemma S.7 imply that

$$\begin{aligned}
&\sup_{x\in[0,1]}\left|n^{-1}\sum\nolimits_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n-x\right)\sigma\left(i/n\right)\varphi_n\left(i/n\right)\varepsilon_i\right| \\
&\leq\sup_{x\in[0,1]}\left|n^{-1}\sum\nolimits_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n-x\right)\sigma\left(i/n\right)\varphi_n\left(i/n\right)Z_{in}\right| \\
&\quad+U_p\left(\tilde{h}^{-1}n^{\beta-1}\rho_n+n^{\beta-1}C_{\varphi,n}\right) \\
&=U_p\left(n^{-1/2}\tilde{h}^{-1/2}\rho_n\log^{1/2}n+n^{\beta-1}C_{\varphi,n}+n^{\beta-1}\tilde{h}^{-1}\rho_n\right),
\end{aligned}$$

which yields Lemma S.8.

S2 Proofs of the Propositions

Proof of Proposition 1 According to Assumption (M4), one has that $\left\|K^{(1)}(x)\right\|_\infty\leq C$. Then for $x\in\mathcal{I}_n$,

$$\begin{aligned}
\sup_{x\in\mathcal{I}_n}\left|\hat{f}(x)-1\right|&=\sup_{x\in\mathcal{I}_n}\left|n^{-1}\sum\nolimits_{i=1}^{n}K_h\left(i/n-x\right)-\int_0^1K_h\left(u-x\right)du\right| \\
&=\sup_{x\in\mathcal{I}_n}\left|\sum\nolimits_{i=1}^{n}\int_{(i-1)/n}^{i/n}\left\{K_h\left(i/n-x\right)-K_h\left(u-x\right)\right\}du\right| \\
&\leq\sum\nolimits_{i=1}^{n}\int_{(i-1)/n}^{i/n}\sup_{x\in\mathcal{I}_n}\left|K_h\left(i/n-x\right)-K_h\left(u-x\right)\right|du \\
&=h^{-1}\sum\nolimits_{i=1}^{n}\int_{(i-1)/n}^{i/n}\sup_{x\in\mathcal{I}_n}\left|K\left(\frac{i/n-x}{h}\right)-K\left(\frac{u-x}{h}\right)\right|du \\
&\leq h^{-2}\sum\nolimits_{i=1}^{n}\int_{(i-1)/n}^{i/n}\left\|K^{(1)}(x)\right\|_\infty\left|u-i/n\right|du\leq Cn^{-1}h^{-2}.
\end{aligned}$$

This completes the proof of Proposition 1.

Proof of Proposition 2 For $x \in \mathcal{I}_n$, $A_n(x)$ in (3.1) can be written as

$$A_n(x) = \sum_{i=1}^{n} \int_{(i-1)/n}^{i/n} K_h(u-x)\{m(u) - m(x)\}\,du + \sum_{i=1}^{n} \int_{(i-1)/n}^{i/n} [K_h(i/n - x)\{m(i/n) - m(x)\} - K_h(u-x)\{m(u) - m(x)\}]\,du.$$

Notice that

$$\begin{aligned}
\left|\sum_{i=1}^{n} \int_{(i-1)/n}^{i/n} K_h(u-x)\{m(u) - m(x)\}\,du\right| &= \left|\int_0^1 K_h(u-x)\{m(u) - m(x)\}\,du\right| \\
&= \left|\int_{-1}^{1} K(v)\{m(x+hv) - m(x)\}\,dv\right| \\
\leq \left|\int_{-1}^{1} K(v)\left\{m^{(1)}(x)hv + \frac{1}{2}m^{(2)}(x)(hv)^2 + \cdots + \frac{1}{(p-1)!}m^{(p-1)}(x)(hv)^{p-1}\right\}dv\right| \\
+ \int_{-1}^{1} \frac{c}{(p-1)!}|K(v)|(hv)^{\theta+p-1}\,dv = \mathcal{O}\left(h^{\theta+p-1}\right). \quad \text{(S2.1)}
\end{aligned}$$

Meanwhile,

$$\begin{aligned}
&\sum_{i=1}^{n} \int_{(i-1)/n}^{i/n} [K_h(i/n - x)\{m(i/n) - m(x)\} - K_h(u-x)\{m(u) - m(x)\}]\,du \\
&= \sum_{i=1}^{n} \int_{(i-1)/n}^{i/n} [K_h(i/n - x)\{m(i/n) - m(x)\} - K_h(i/n-x)\{m(u) - m(x)\}]\,du \\
&+ \sum_{i=1}^{n} \int_{(i-1)/n}^{i/n} [K_h(i/n - x)\{m(u) - m(x)\} - K_h(u-x)\{m(u) - m(x)\}]\,du.
\end{aligned} \quad \text{(S2.2)}$$

Thus, the first term in Equation (S2.2) equals

$$\begin{aligned}
&\left|\sum_{i=1}^{n} \int_{(i-1)/n}^{i/n} K_h(i/n - x)\{m(i/n) - m(u)\}\,du\right| \\
&\leq h^{-1}\|K\|_\infty \left\|m^{(1)}(x)\right\|_\infty \sum_{i=1}^{n} \int_{(i-1)/n}^{i/n} |i/n - u|\,du = \mathcal{O}\left(n^{-1}h^{-1}\right) \quad \text{(S2.3)}
\end{aligned}$$

and the second term in Equation (S2.2) equals

$$\left|\sum_{i=1}^{n}\int_{(i-1)/n}^{i/n}\left\{K_h\left(i/n-x\right)-K_h\left(u-x\right)\right\}\left\{m\left(u\right)-m(x)\right\}du\right|$$
$$\leq\sum_{i=1}^{n}\int_{(i-1)/n}^{i/n}\left|\left\{K_h\left(i/n-x\right)-K_h\left(u-x\right)\right\}\left\{m\left(u\right)-m(x)\right\}\right|du$$
$$\leq C\left\|m^{(1)}\right\|_\infty\left\|K^{(1)}\right\|_\infty h\sum_{i=1}^{n}\int_{(i-1)/n}^{i/n}h^{-2}\left|i/n-u\right|du=\mathcal{O}\left(n^{-1}h^{-1}\right). \tag{S2.4}$$

Collecting (S2.1), (S2.2), (S2.3) and (S2.4) establishes Proposition 2.

Proof of Proposition 3(a) Equations (3.2) and (3.3) imply that $\sup_{x\in[0,1]}|B_n(x) -B_{n1}(x)|$ can be written as

$$\sup_{x\in[0,1]}\left|n^{-1}\sum\nolimits_{i=1}^{n}K_h\left(x-i/n\right)\sigma\left(i/n\right)\left\{S_i-W_{i,n}+W_{i-1,n}-S_{i-1}\right\}\right|$$
$$\leq\sup_{x\in[0,1]}\left|\Delta_n(x)\right|+\sup_{x\in[0,1]}\left|n^{-1}K_h\left(x-1\right)\sigma\left(n/n\right)\left\{S_n-W_{n,n}\right\}\right|, \tag{S2.5}$$

in which $\Delta_n(x)$ equals

$$n^{-1}\sum\nolimits_{i=1}^{n-1}\left\{K_h\left(x-i/n\right)\sigma\left(i/n\right)-K_h\left(x-\left(i+1\right)/n\right)\sigma\left(\left(i+1\right)/n\right)\right\}\left(S_i-W_{i,n}\right),$$

and $\{S_i\}_{i=1}^n$ and $\{W_{i,n}\}_{i=1}^n$ are in Equation (S1.1) with $S_0=0, W_{0,n}=0$. Under Assumption (M2), one obtains that $\max_{1\leq i\leq n-1}\left|\sigma\left(\left(i+1\right)/n\right)-\sigma\left(i/n\right)\right| =\mathcal{O}\left(n^{-1}\right)$. Thus, for n large enough,

$$\max_{1\leq i\leq n-1}\sup_{x\in[0,1]}\left|\left\{K_h\left(x-i/n\right)\sigma\left(i/n\right)-K_h\left(x-\left(i+1\right)/n\right)\sigma\left(\left(i+1\right)/n\right)\right\}\right|$$
$$=\max_{1\leq i\leq n-1}\sup_{x\in[0,1]}\left|\sigma\left(i/n\right)\left\{K_h\left(x-i/n\right)-K_h\left(x-\left(i+1\right)/n\right)+\mathcal{O}\left(n^{-1}h^{-1}\right)\right\}\right|$$
$$\leq C_\sigma\left\|K^{(1)}\right\|_\infty h^{-2}n^{-1}=\mathcal{O}\left(n^{-1}h^{-2}\right),$$

which along with Assumption (M3) implies that

$$\begin{aligned}\sup_{x\in[0,1]}|\Delta_n(x)| &\le n^{-1}([2nh]+1)\max_{1\le i\le n-1}|S_i-W_{i,n}|\sup_{x\in[0,1]}\max_{1\le i\le n-1}\\ &\quad |K_h(x-i/n)\sigma(i/n)-K_h(x-(i+1)/n)\sigma((i+1)/n)|\\ &= U_p\left(n^{\beta-1}h^{-1}\right). \qquad \text{(S2.6)}\end{aligned}$$

Meanwhile, according to Lemma S.3,

$$\begin{aligned}\left|n^{-1}K_h(x-1)\sigma(n/n)\{S_n-W_{n,n}\}\right| &\le h^{-1}C_\sigma\|K\|_\infty n^{-1}|S_n-W_{n,n}|\\ &= \mathcal{O}_{a.s.}\left(n^{\beta-1}h^{-1}\right). \qquad \text{(S2.7)}\end{aligned}$$

Then Equations (S2.5), (S2.6) and (S2.7) imply Proposition 3(a).

Proof of Proposition 3(b) According to Equations (3.3) and (3.4), $B_{n1}(x)-B_{n2}(x)$ is the Gaussian process with mean 0 and variance

$$\begin{aligned}&\mathrm{E}\left\{n^{-1}\sum\nolimits_{i=1}^n K_h(i/n-x)\{\sigma(i/n)-\sigma(x)\}Z_{in}\right\}^2\\ &= n^{-2}\sum\nolimits_{i=1}^n K_h^2(i/n-x)\{\sigma(i/n)-\sigma(x)\}^2\\ &\le n^{-2}h^{-2}\|K\|_\infty^2\left\|\sigma^{(1)}(x)\right\|_\infty^2 h^2([2nh]+1)\le Cn^{-1}h.\end{aligned}$$

Using the tail property of the normal distribution and applying the discretization method as in the proof of Lemma S.7, one immediately obtains that

$$\sup_{x\in[0,1]}|B_{n1}(x)-B_{n2}(x)|=\mathcal{O}_p\left(n^{-1/2}h^{1/2}\log^{1/2}n\right),$$

which completes the proof of Proposition 3(b).

Proof of Proposition 3(c) The $B_{n2}(x)$ in (3.4) can be written as the Gaussian process $n^{-1/2}\sum_{i=1}^n K_h(x-i/n)\sigma(x)\{W_{i,n}-W_{i-1,n}\}$ where $\{W_{i,n}\}_{i=1}^n$ are in Equation (S1.1) with $S_0=0, W_{0,l}=0$. Meanwhile note that, under Assumption (M4), $K_h(x-u)=0$ for $x\in\mathcal{I}_n=[h,1-h]$ and

$u \in (-\infty, 0] \cup [1, +\infty)$. Thus, for $x \in \mathcal{I}_n$,

$$\begin{aligned}
B_{n2}(x) &= n^{-1/2} \sum\nolimits_{i=1}^{n} K_h\left(x - i/n\right) \sigma(x) \left\{W_{i,n} - W_{i-1,n}\right\} \\
&= n^{-1/2} \sigma(x) \sum\nolimits_{i=1}^{n} \int_{(i-1)/n}^{i/n} K_h\left(x-u\right) dW_n\left(u\right) \\
&+ n^{-1/2} \sigma(x) \sum\nolimits_{i=1}^{n} \int_{(i-1)/n}^{i/n} \left[K_h\left(x - i/n\right) - K_h\left(x-u\right)\right] dW_u\left(u\right) \\
&= \int K_h\left(x-u\right) dW_n\left(u\right) + n^{-1/2} \sigma\left(x\right) \sum\nolimits_{i=1}^{n} \int_{(i-1)/n}^{i/n} \left\{K_h\left(x - i/n\right)\right. \\
&\left. - K_h\left(x-u\right)\right\} dW_u\left(u\right) = B_{n3}(x) + \kappa_n(x), \qquad \text{(S2.8)}
\end{aligned}$$

where $\kappa_n(x) = n^{-1/2}\sigma(x) \sum_{i=1}^{n} \int_{(i-1)/n}^{i/n} \left\{K_h\left(x - i/n\right) - K_h\left(x-u\right)\right\} dW_u\left(u\right)$ is a Gaussian process with mean 0 and variance

$$\begin{aligned}
\mathrm{E}\,\kappa_n^2(x) &= \mathrm{E}\left[n^{-1/2}\sigma(x) \sum_{i=1}^{n} \int_{(i-1)/n}^{i/n} \left\{K_h\left(x - i/n\right) - K_h\left(x-u\right)\right\} dW_n\left(u\right)\right]^2 \\
&= n^{-1}\sigma^2(x) \sum_{i=1}^{n} \mathrm{E}\left[\int_{(i-1)/n}^{i/n} \left\{K_h\left(x - i/n\right) - K_h\left(x-u\right)\right\} dW_n\left(u\right)\right]^2 \\
&= n^{-1}\sigma^2(x) \sum_{i=1}^{n} \int_{(i-1)/n}^{i/n} \left\{K_h\left(x - i/n\right) - K_h\left(x-u\right)\right\}^2 du.
\end{aligned}$$

Thus the variance of $\kappa_n(x)$ is bounded by

$$C_\sigma^2 n^{-1} h^{-4} \sum_{i=1}^{n} \int_{(i-1)/n}^{i/n} \left\|K^{(1)}\right\|_\infty^2 \left|u - i/n\right|^2 du \le C_\sigma^2 \left\|K^{(1)}\right\|_\infty^2 n^{-3} h^{-4} \le C n^{-3} h^{-4}.$$

Similarly, using the tail property of the normal distribution and applying the discretization method, one obtains that

$$\sup_{x \in [0,1]} \left|\kappa_n(x)\right| = \mathcal{O}_{a.s}\left(n^{-3/2} h^{-2} \log^{1/2} n\right),$$

which together with (S2.8) implies Proposition 3(c).

Proof of Proposition 3(d) The $B_{n3}(x)$ in (3.5) is the Gaussian process with mean 0 and variance

$$\begin{aligned}
\mathrm{E}\, B_{n3}^2(x) &= n^{-1}\sigma^2(x)\,\mathrm{E}\left\{\int K_h\left(x-u\right)dW_n\left(u\right)\right\}^2 \\
&= n^{-1}\sigma^2(x)\int K_h^2\left(x-u\right)du = n^{-1}h^{-1}\sigma^2(x)\int_{-1}^{1} K^2\left(u\right)du.
\end{aligned}$$

Once again using the tail property of the normal distribution and applying the discretization method, one obtains Proposition 3(d).

Proof of Proposition 5 Obviously, $I_1(x)$ in (3.10) is bounded by

$$\begin{aligned}
& n^{-1}\sum\nolimits_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n-x\right)\left[2\left\{m\left(i/n\right)-\tilde{m}_p\left(i/n\right)\right\}^2+2\,\tilde{\varepsilon}_p^2\left(i/n\right)\right] \\
& \leq 2n^{-1}\sum\nolimits_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n-x\right)\left\{\left\|m(x)-\tilde{m}_p\left(x\right)\right\|_\infty^2+\ \left\|\tilde{\varepsilon}_p(x)\right\|_\infty^2\right\} \\
& \leq U_p\left(N^{-2p}+n^{-1}N\right),
\end{aligned}$$

completing the proof of Proposition 5.

Proof of Proposition 6 From Equation (3.7), one knows that $I_2(x)$ in (3.11) equals $2n^{-1}\sum_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n-x\right)\sigma\left(i/n\right)\varepsilon_i\sum_{j=1-p}^{N}\tilde{a}_{j,p}B_{j,p}(i/n)$. Hence the Cauchy- Schwartz inequality implies that

$$|I_2(x)|\leq 2\left[\sum\nolimits_{j=1-p}^{N}\tilde{a}_{j,p}^2\sum\nolimits_{j=1-p}^{N}\left\{n^{-1}\sum\nolimits_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n-x\right)\sigma\left(i/n\right)\varepsilon_i B_{j,p}\left(i/n\right)\right\}^2\right]^{1/2}. \tag{S2.9}$$

According to Equation (S1.2) in Lemma S.5

$$c\left\|\sum\nolimits_{j=1-p}^{N}\tilde{a}_{j,p}B_{j,p}(x)\right\|_2^2\leq\sum\nolimits_{j=1-p}^{N}\tilde{a}_{j,p}^2\leq C\left\|\sum\nolimits_{j=1-p}^{N}\tilde{a}_{j,p}B_{j,p}(x)\right\|_2^2.$$

Hence applying Equation (3.7) and Lemma S.6, one obtains that

$$\sum\nolimits_{j=1-p}^{N}\tilde{a}_{j,p}^2=\mathcal{O}_p\left(n^{-1}N\right). \tag{S2.10}$$

Applying Lemma **S.8** with $\varphi_n(x)=B_{j,p}(x)$, $C_{\varphi,n}=\mathcal{O}\left(N^{3/2}\right)$ and $\rho_n=$

$\mathcal{O}\left(N^{1/2}\right)$, one has that

$$\sup_{x\in[0,1]}\left|n^{-1}\sum\nolimits_{j=1-p}^{N}\tilde{K}_{\tilde{h}}\left(i/n-x\right)\sigma\left(i/n\right)\varepsilon_i B_{j,p}\left(i/n\right)\right|$$
$$=\mathcal{O}_p\left(n^{-1/2}\tilde{h}^{-1/2}N^{1/2}\log^{1/2}n+\tilde{h}^{-1}n^{\beta-1}N^{1/2}+n^{\beta-1}N^{3/2}\right). \tag{S2.11}$$

Putting (**S2.10**) and (**S2.11**) together one obtains that

$$\sup_{x\in[0,1]}\left[\sum\nolimits_{j=1-p}^{N}\tilde{a}_{j,p}^2\sum\nolimits_{j=1-p}^{N}\left\{n^{-1}\sum\nolimits_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n-x\right)\sigma\left(i/n\right)\varepsilon_i B_{j,p}\left(i/n\right)\right\}^2\right]^{1/2}$$
$$=\mathcal{O}_p\left(n^{-1}\tilde{h}^{-1/2}N^{3/2}\log^{1/2}n+\tilde{h}^{-1}n^{\beta-3/2}N^{3/2}+n^{\beta-3/2}N^{5/2}\right),$$

which together with (S2.9) implies Proposition 6.

Proof of Proposition 7 For I_3 in (3.12), one obtains that

$$\begin{aligned}|I_3(x)| &= \left|n^{-1}\sum\nolimits_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n-x\right)\left\{m\left(i/n\right)-\tilde{m}_p\left(i/n\right)\right\}\sigma\left(i/n\right)\varepsilon_i\right| \\ &\le \left|n^{-1}\sum\nolimits_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n-x\right)\left\{m\left(i/n\right)-g\left(i/n\right)\right\}\sigma\left(i/n\right)\varepsilon_i\right| + \\ &\quad \left|n^{-1}\sum\nolimits_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n-x\right)\left\{g\left(i/n\right)-\tilde{m}_p\left(i/n\right)\right\}\sigma\left(i/n\right)\varepsilon_i\right| \end{aligned} \tag{S2.12}$$

in which $g\in\mathcal{H}_N^{(p-2)}$ satisfies $\|g-m\|_\infty\le C_pN^{-p}$, $g-m\in\mathrm{Lip}\left(\left[0,1\right],CN^{1-p}\right)$. Applying Lemma S.8 with $\varphi_n(x)=m(x)-g(x)$, one obtains that

$$\sup_{x\in[0,1]}\left|n^{-1}\sum\nolimits_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n-x\right)\left\{m\left(i/n\right)-g\left(i/n\right)\right\}\sigma\left(i/n\right)\varepsilon_i\right|$$
$$=U_p\left(n^{-1/2}\tilde{h}^{-1/2}N^{1-p}\log^{1/2}n+n^{\beta-1}N^{1-p}+n^{\beta-1}\tilde{h}^{-1}N^{-p}\right). \tag{S2.13}$$

Since $g,\tilde{m}_p\in\mathcal{H}_N^{(p-2)}$, one can write $g(x)-\tilde{m}_p(x)=\sum\nolimits_{J=1-p}^{N}\pi_{J,p}B_{J,p}(x)$. According to Lemmas S.4 and S.5, we have

$$\sum\nolimits_{J=1-p}^{N}\pi_{J,p}^2\le C\left\|\sum\nolimits_{J=1-p}^{N}\pi_{J,p}B_{J,p}(x)\right\|_2^2\le C\left\|g(x)-\tilde{m}_p(x)\right\|_\infty=U_p\left(N^{-2p}\right). \tag{S2.14}$$

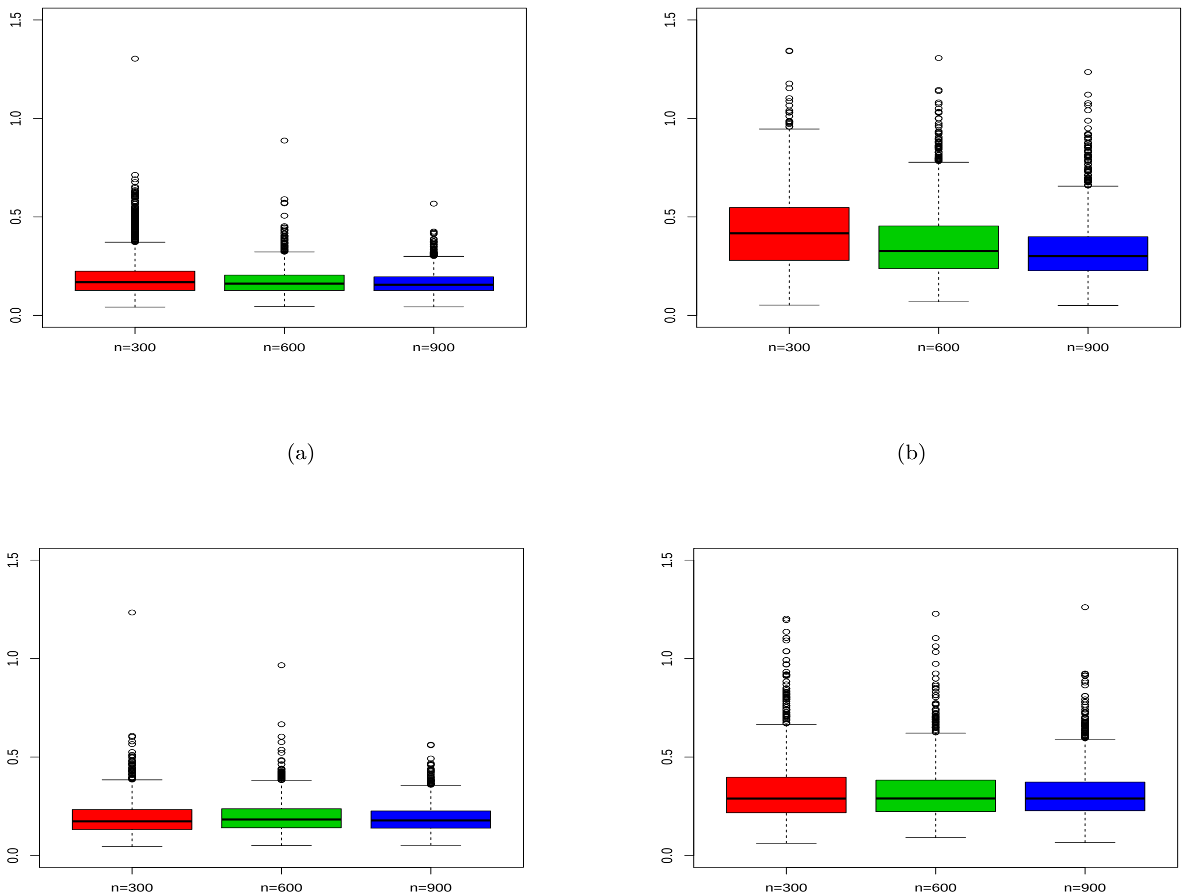
Equations (S2.11) and (S2.14) imply that

$$\sup_{x\in[0,1]}\left|n^{-1}\sum\nolimits_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n-x\right)\left\{g\left(i/n\right)-\tilde{m}_p\left(i/n\right)\right\}\sigma\left(i/n\right)\varepsilon_i\right|$$
$$\leq\sup_{x\in[0,1]}\left[\sum\nolimits_{J=1-p}^{N}\pi_{J,p}^2\sum\nolimits_{J=1-p}^{N}\left\{n^{-1}\sum\nolimits_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n-x\right)B_{J,p}\left(i/n\right)\sigma\left(i/n\right)\varepsilon_i\right\}^2\right]^{1/2}$$
$$=U_p\left(n^{-1/2}\tilde{h}^{-1/2}N^{1-p}\log^{1/2}n+\tilde{h}^{-1}n^{\beta-1}N^{1-p}+n^{\beta-1}N^{2-p}\right),$$

which together with (S2.12) and (S2.13) implies Proposition 7.

S3 Figures

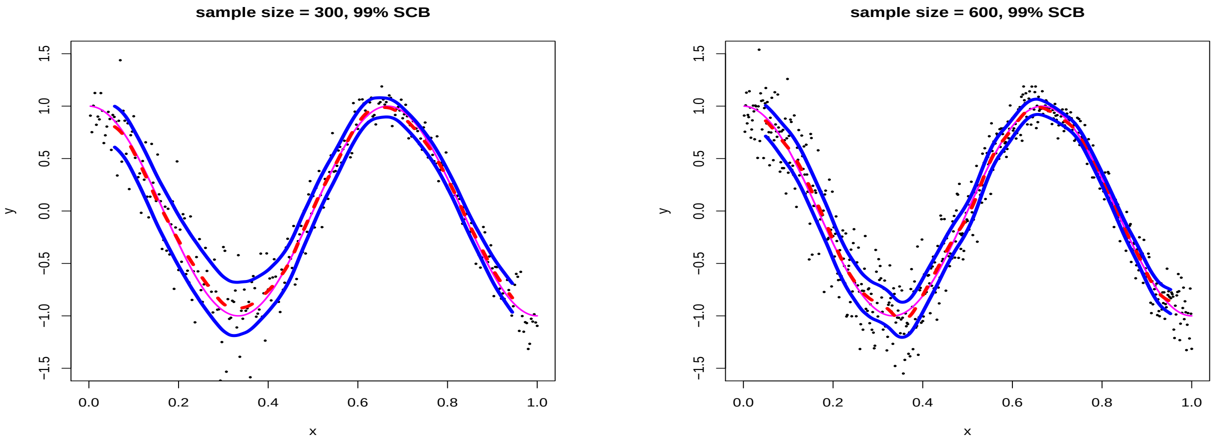
This section shows some figures for the simulated data examples with the noise ε following a t-distribution and the strata pressure data examples.



(a) (b)

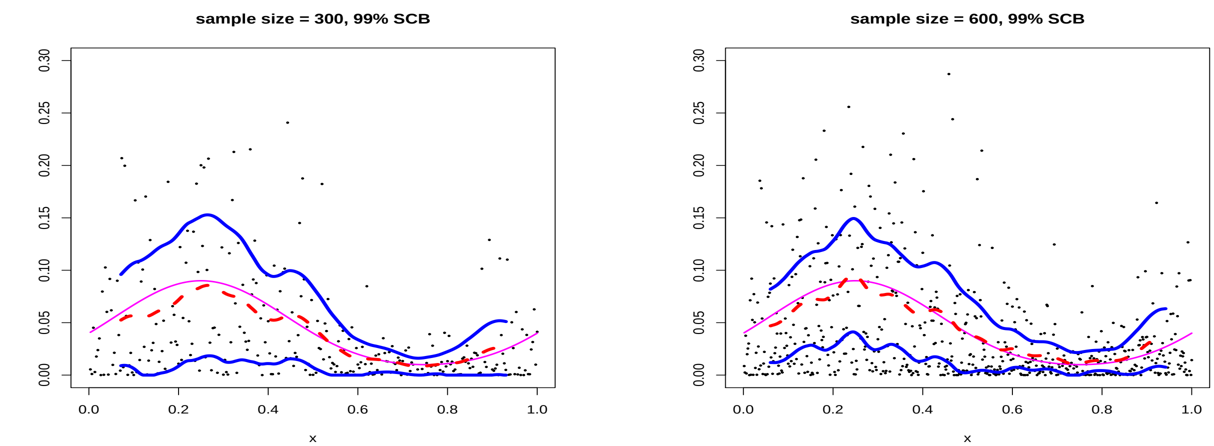
(c) (d)

Figure S.1: Boxplots of $\triangle_n = \sqrt{n}\max_{j=1}^{400}\left|\tilde{\sigma}^2_{\mathrm{K}}\left(x_j\right) - \hat{\sigma}^2_{\mathrm{SK}}\left(x_j\right)\right|$ in which x_j are the equally spaced points on $\tilde{\mathcal{I}}_n$ over 2000 replications with $\varepsilon \sim \sqrt{0.8} * t_{10}$: (a) Case 1; (b) Case 2; (c) Case 3; (d) Case 4.



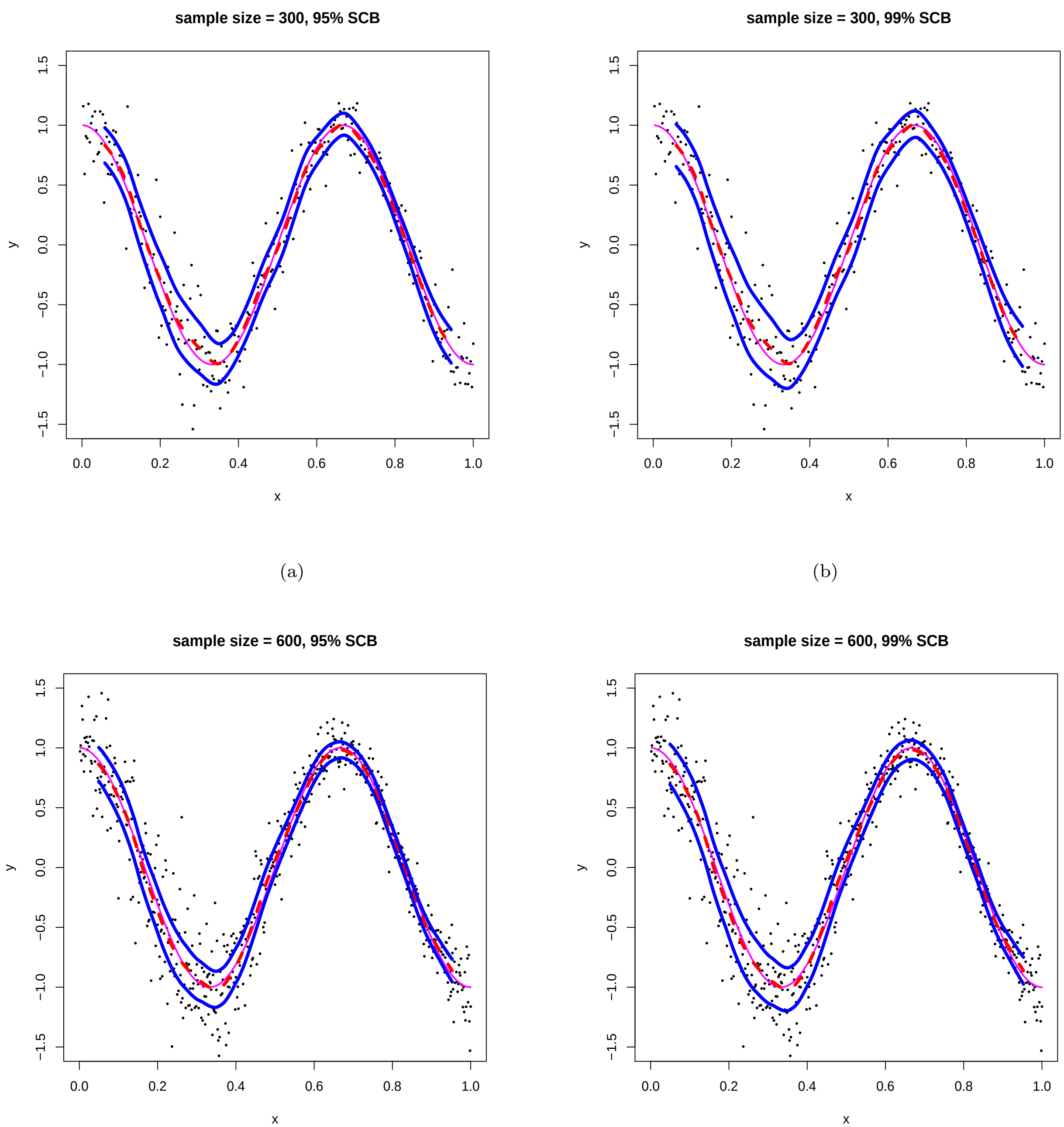
(a) (b)

Figure S.2: Plots of 99% SCB (thick solid) for $m(x)$ (solid) and the estimator $\hat{m}(x)$ (dashed) in Case 1 with $\varepsilon \sim N(0,1)$ and $n = 300, 600$ respectively.



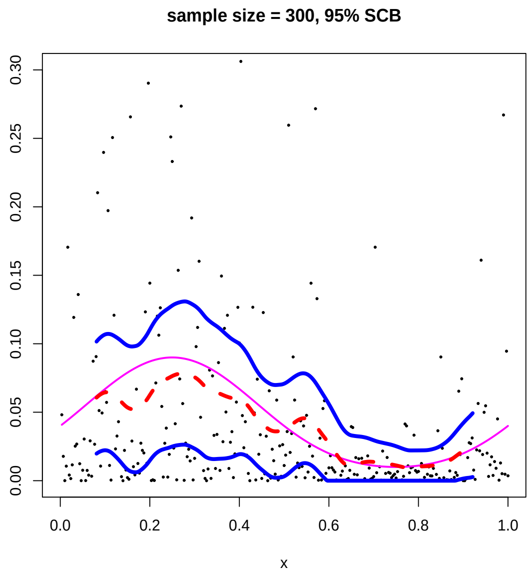
(a) (b)

Figure S.3: Plots of 99% SCB (thick solid) for $\sigma^2(x)$ (solid) and the estimator $\hat{\sigma}^2_{\mathrm{SK}}(x)$ (dashed) in Case 1 with $\varepsilon \sim N(0,1)$ and $n = 300, 600$ respectively.

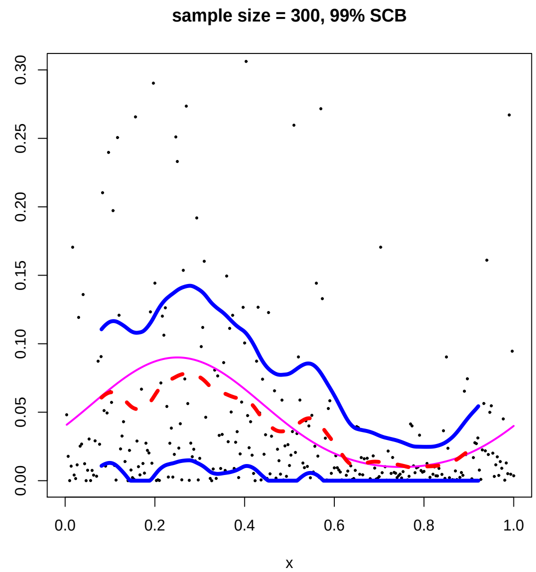


(a) (b) (c) (d)

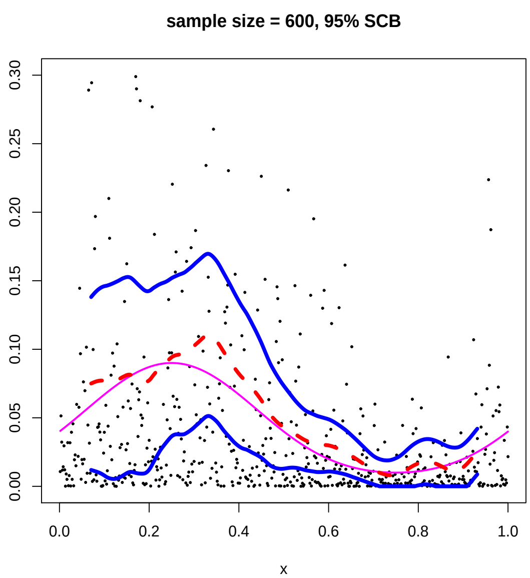
Figure S.4: Plots of SCB (thick solid) for $m(x)$ (solid) in Case 1 with $\varepsilon \sim \sqrt{0.8} * t_{10}$ which is computed according to (4.2) and the estimator $\hat{m}(x)$ (dashed).



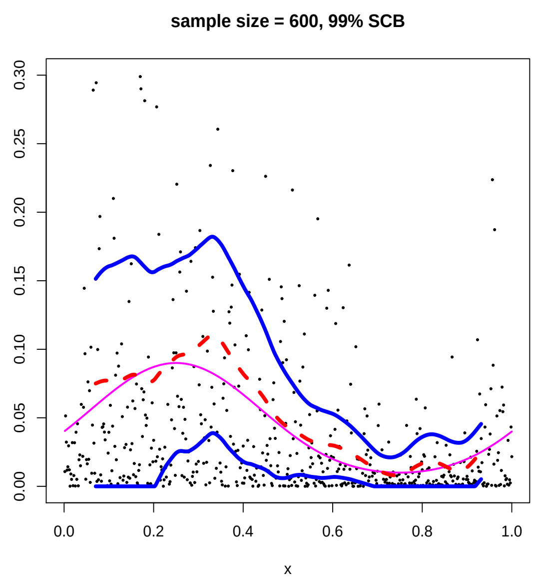
(a)



(b)



(c)



(d)

Figure S.5: Plots of SCB (thick solid) for $\sigma^2(x)$ (solid) in Case 1 with $\varepsilon \sim \sqrt{0.8} * t_{10}$ and the estimator $\hat{\sigma}^2_{\mathrm{SK}}(x)$ (dashed).

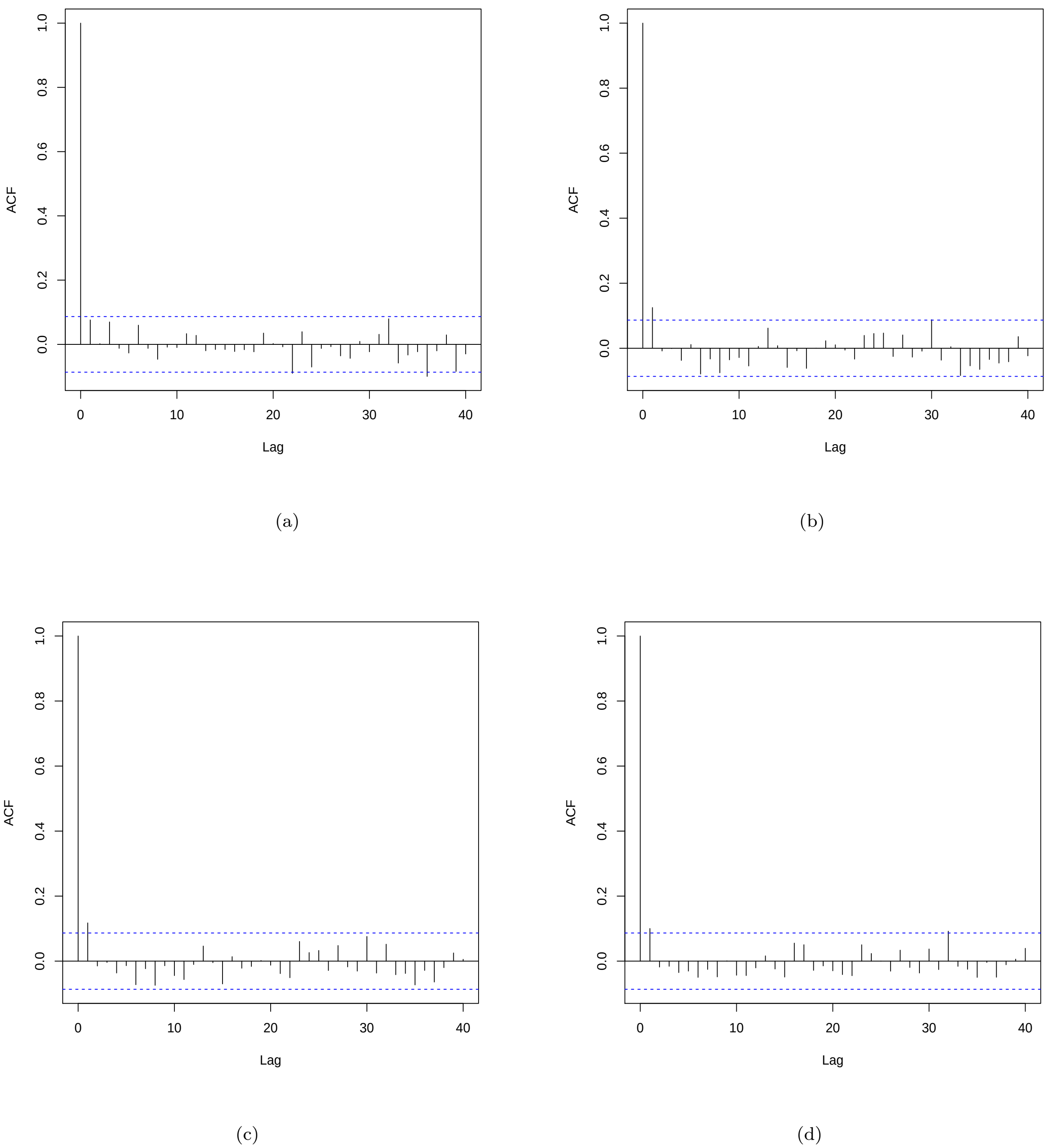
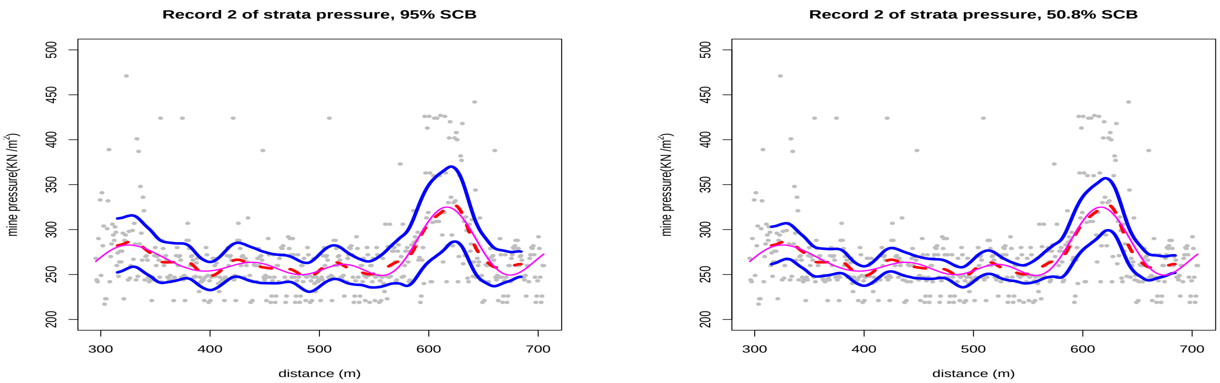
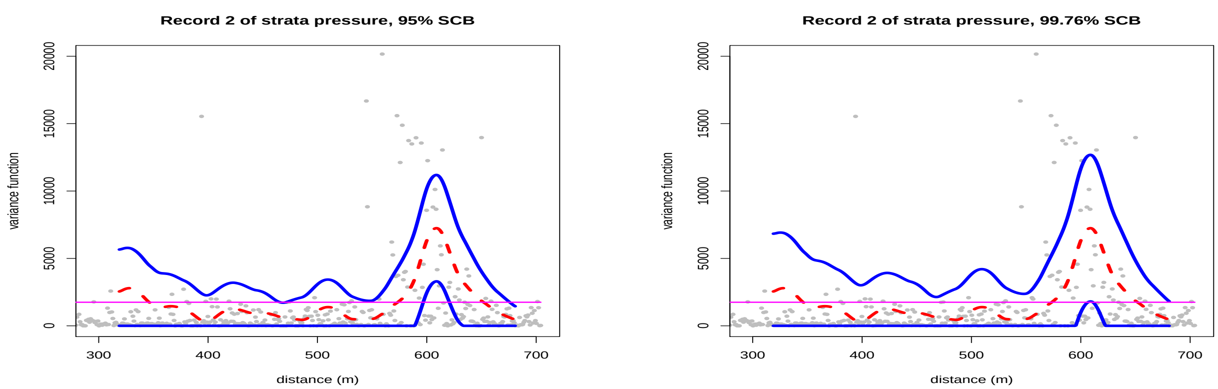


Figure S.6: For record 1, plots of the acfs of (a) $\{\hat{\varepsilon}_i\}_{i=1}^n$, (b) $\{|\hat{\varepsilon}_i|\}_{i=1}^n$, (c) $\{\hat{\varepsilon}_i^2\}_{i=1}^n$, (d) $\{\hat{\varepsilon}_i^4\}_{i=1}^n$.



(a) (b)

Figure S.7: For record 2, plots of the null hypothesis curve of $m(x) = a_0 + \sum_{k=1}^{5}\{a_k \sin(k\omega x) + b_k \cos(k\omega x)\}$ (solid), kernel estimator $\hat{m}(x)$ (dashed), SCB (thick solid) for $m(x)$ with (a) $\alpha = 0.05$ and (b) $\alpha = 0.492$.



(a) (b)

Figure S.8: For record 2, plots of the null hypothesis curve of $\hat{\sigma}^2 = n^{-1}\sum_{i=1}^{n}\hat{e}_i^2$ (solid), SCB (thick solid) for $\sigma^2(x)$ and the spline-kernel estimator $\hat{\sigma}^2_{\mathrm{SK}}(x)$ (dashed) with (a) $\alpha = 0.05$ and (b) $\alpha = 0.0024$.