SIMULTANEOUS CONFIDENCE BANDS FOR MEAN AND VARIANCE FUNCTIONS BASED ON DETERMINISTIC DESIGN

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Supplementary Materials

This supplement contains proofs for Lemmas A.2, A.6, A.7, A.8 and Proposition 1 and some figures for the simulated and real data examples.

S1 Proofs for some technical lemmas

In this section, we provide the detailed proofs for Lemmas A.2, A.6, A.7 and A.8.

Proof of Lemma A.2. Under Assumption (M3') that $\mathrm{E}\left|\varepsilon_1\right|^{2+\eta} < +\infty$ for some $\eta > 2/\beta - 2, \beta \in (0, 1/2 - 1/(4\theta + 4p - 2))$, hence $\eta > 2$. Now let $H(x) = x^{2+\eta}$ which is increasing for $x > 0$. If one sets $x_0 = 3$, then for $x \geq x_0, 0 < \delta < 1$, $x^{-3-\delta}H(x)$ is monotone increasing, $x^{-1}\log H(x)$ is monotone decreasing, hence all conditions of Lemma A.1 are met. Let $K_n = n^{1/(2+\eta)}, \gamma = (\eta + 2)\beta - 1 > 1, t = t_n = n^{\beta}$, then Lemma A.1 entails that there exist constants $C_1, C_2, a > 0$ depending only on the distributions of ε_1 and independent standard normal variables $\{Z_{in}\}_{i=1}^n$ such that for $t_n > K_n, t_n^2/\log H(t_n) < C_1 n$, $n/H(at_n) = (aC_1)^{-2-\eta} n^{1-(2+\eta)\beta} = a^{-2-\eta} n^{-\gamma}$, let $C_0 = C_2 (C_1 a)^{-2-\eta}$, then

$$P\left\{\max_{1\le l\le n} |S_l - W_{l,n}| > n^{\beta}\right\} \le C_0 n^{-\gamma}, S_l = \sum_{i=1}^{l} \varepsilon_i, W_{l,n} = \sum_{i=1}^{l} Z_{in}, \quad \text{(S1.1)}$$

which implies Lemma A.2.

Proof of Lemma A.6. According to Equation (3.9), $\tilde{\varepsilon}_p(x)$ can be written as

$$\{B_{J,p}(x)\}_{J=1-p}^{N} \left(n^{-1}\mathbf{B}^T\mathbf{B}\right)^{-1} \left\{n^{-1}\sum\nolimits_{i=1}^{n} B_{J,p}(i/n)\,\sigma(i/n)\,\varepsilon_i\right\}_{J=1-p}^{N}.$$

By applying Equation (A.1) in Lemma A.5, $\tilde{\varepsilon}_p(x)$ is bounded by

$$C\left\|\{B_{J,p}(x)\}_{j=1-p}^{N}\right\| \left\|\left\{n^{-1}\sum\nolimits_{i=1}^{n} B_{J,p}(i/n)\,\sigma(i/n)\,\varepsilon_i\right\}_{J=1-p}^{N}\right\|.$$

The definition of $\{B_{J,p}(x)\}_{j=1-p}^{N}$ and the simple fact that $\|b_{J,p}(x)\|_2 \ge cN^{-1/2}$, $1-p \le J \le N$, for some constant c imply that $\sup_{x\in[0,1]}\left\|\{B_{J,p}(x)\}_{J=1-p}^{N}\right\| = \mathcal{O}\left(N^{1/2}\right)$. Hence,

$$\sup_{x\in[0,1]} |\tilde{\varepsilon}_p(x)| = O\left(N^{1/2}\right)\left\|\left\{n^{-1}\sum\nolimits_{i=1}^{n} B_{J,p}(i/n)\,\sigma(i/n)\,\varepsilon_i\right\}_{J=1-p}^{N}\right\|. \quad \text{(S1.2)}$$

Let $[x]$ represent the integer part of x. Notice that

$$\begin{aligned}
&\mathrm{E}\left\|\left\{n^{-1}\sum\nolimits_{i=1}^{n} B_{J,p}(i/n)\,\sigma(i/n)\,\varepsilon_i\right\}_{J=1-p}^{N}\right\|^2 \\
&= \mathrm{E}\sum\nolimits_{J=1-p}^{N}\left\{n^{-1}\sum\nolimits_{i=1}^{n} B_{J,p}(i/n)\,\sigma(i/n)\,\varepsilon_i\right\}^2 \\
&= \sum\nolimits_{J=1-p}^{N} n^{-2}\sum\nolimits_{i=1}^{n} B_{J,p}^2(i/n)\,\sigma^2(i/n) \\
&\le C_\sigma^2 (N+p)\, n^{-2} C\left(\left[pnN^{-1}\right]+1\right) = \mathcal{O}\left(n^{-1}\right).
\end{aligned}$$

Thus, $\left\|\{n^{-1}\sum_{i=1}^{n} B_{J,p}\left(i/n\right)\sigma\left(i/n\right)\varepsilon_i\}_{J=1-p}^{N}\right\| = \mathcal{O}_p\left(n^{-1/2}\right)$ which together with (S1.2) implies that

$$\sup_{x\in[0,1]}\left|\tilde{\varepsilon}_p\left(x\right)\right| = \mathcal{O}_p\left(n^{-1/2}N^{1/2}\right).$$

Proof of Lemma A.7. Let $\Lambda_n(x) = n^{-1}\sum_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n - x\right)\sigma\left(i/n\right)\varphi_n\left(i/n\right)Z_{in}$. Obviously, $\Lambda_n(x)$ is the Gaussian process with mean 0 and variance

$$\begin{aligned}
\mathrm{E}\left\{\Lambda_n(x)\right\}^2 &= n^{-2}\sum\nolimits_{i=1}^{n}\tilde{K}_{\tilde{h}}^2\left(i/n - x\right)\sigma^2\left(i/n\right)\varphi_n^2\left(i/n\right)\\
&\le C_\sigma^2\rho_n^2 n^{-2}\sum\nolimits_{i=1}^{n}\tilde{K}_{\tilde{h}}^2\left(i/n - x\right)\\
&\le C_\sigma^2\rho_n^2 n^{-2}\tilde{h}^{-2}\left\|\tilde{K}\right\|_\infty^2\left(\left[2n\tilde{h}\right]+1\right)\\
&\le C_\sigma^2\rho_n^2\left\|\tilde{K}\right\|_\infty n^{-1}\tilde{h}^{-1}.
\end{aligned}$$

In the following, we use the well-known tail property of the normal distribution, i.e., $1-\Phi(x)\le\phi(x)/x$ for $x\ge 0$, in which $\Phi(x)$ and $\phi(x)$ are the cumulative distribution function and the density function of the standard normal respectively. Hence there exists some $c>0$ such that $1-\Phi(x)\le c\phi(x)$ for large x. Take $x=\delta_n=\sqrt{16\log n}$, and hence there exists a constant c such that for a large enough n,

$$P\left\{\left|\Lambda_n(x)\right|\ge C\rho_n n^{-1/2}\tilde{h}^{-1/2}\delta_n\right\}\le c\exp\left\{-\delta_n^2/2\right\} = cn^{-8}.$$

Divide the interval $[0,1]$ into $M_n = n^4$ equally spaced intervals with disjoint endpoints $0 = x_0 < x_1 < \cdots < x_{M_n} = 1$, so that the consecutive endpoints make a total of M_n subintervals with length M_n^{-1}. One immediately obtains that

$$P\left\{\max_{0\leq k\leq M_n}\left|\Lambda_n\left(x_k\right)\right|\geq C\rho_n n^{-1/2}\tilde{h}^{-1/2}\delta_n\right\}$$
$$\leq\sum_{k=0}^{M_n}P\left\{\left|\Lambda_n\left(x_k\right)\right|\geq C\rho_n n^{-1/2}\tilde{h}^{-1/2}\delta_n\right\}$$
$$\leq\sum_{k=0}^{M_n}cn^{-8}=cn^{-4}.$$

Thus, $\sum_{n=1}^{\infty}P\left\{\max_{0\leq k\leq M_n}\left|\Lambda_n\left(x_k\right)\right|\geq C\rho_n n^{-1/2}\tilde{h}^{-1/2}\delta_n\right\}<+\infty$. The Borrel-Cantelli Lemma implies

$$\max_{0\leq k\leq M_n}\left|\Lambda_n\left(x_k\right)\right|=\mathcal{O}_{a.s}\left(n^{-1/2}\tilde{h}^{-1/2}\rho_n\log^{1/2}n\right). \tag{S1.3}$$

Taking the supremum over the whole interval $x\in[0,1]$, one obtains that

$$\max_{x\in[0,1]}\left|\Lambda_n(x)\right|\leq\max_{0\leq k\leq M_n}\left|\Lambda_n\left(x\right)-\Lambda_n\left(x_k\right)\right|+\max_{0\leq k\leq M_n-1}\sup_{x\in[x_k,x_{k+1}]}\left|\Lambda_n\left(x_k\right)\right|. \tag{S1.4}$$

Meanwhile, $\max_{0\leq k\leq M_n-1}\sup_{x\in[x_k,x_{k+1}]}\left|\Lambda_n(x)-\Lambda_n\left(x_k\right)\right|$ is bounded by

$$\max_{0\leq k\leq M_n-1}\sup_{x\in[x_k,x_{k+1}]}n^{-1}\sum_{i=1}^{n}\left|\tilde{K}_{\tilde{h}}\left(x-i/n\right)-\tilde{K}_{\tilde{h}}\left(x_k-i/n\right)\right|\rho_n\sigma\left(i/n\right)\left|Z_{in}\right|$$
$$\leq C_\sigma n^{-1}\tilde{h}^{-2}\left\|\tilde{K}^{(1)}\right\|_\infty\rho_n M_n^{-1}\sum_{i=1}^{n}\left|Z_{in}\right|=o_p\left(n^{-2}\rho_n\right). \tag{S1.5}$$

Consequently, Equations (S1.3), (S1.4) and (S1.5) establish Lemma A.7.

Proof of Lemma A.8 According to Equation (S1.1), it is obvious that

$$\varepsilon_i=S_i-S_{i-1},Z_{in}=W_{i,n}-W_{i-1,n},1\leq i\leq n,$$

with $S_0=0,W_{0,n}=0$. Denote $\Upsilon_n\left(i,x\right)=\tilde{K}_{\tilde{h}}\left(i/n-x\right)\sigma\left(i/n\right)\varphi_n\left(i/n\right)$. Therefore

$$n^{-1}\sum\nolimits_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n-x\right)\sigma\left(i/n\right)\varphi_n\left(i/n\right)\varepsilon_i=n^{-1}\sum\nolimits_{i=1}^{n}\Upsilon_n\left(i,x\right)\left(S_i-S_{i-1}\right),$$

and

$$\begin{aligned}
&n^{-1} \sum\nolimits_{i=1}^{n} \tilde{K}_{\tilde{h}}\left(i/n-x\right)\sigma\left(i/n\right)\varphi_n\left(i/n\right)Z_{in} \\
&= n^{-1} \sum\nolimits_{i=1}^{n} \Upsilon_n\left(i,x\right)\left(W_{i,n}-W_{i-1,n}\right).
\end{aligned}$$

Thus, $n^{-1}\sum_{i=1}^{n} \tilde{K}_{\tilde{h}}\left(i/n-x\right)\sigma\left(i/n\right)\varphi_n\left(i/n\right)\varepsilon_i - n^{-1}\sum_{i=1}^{n} \tilde{K}_{\tilde{h}}\left(i/n-x\right)\sigma\left(i/n\right)$ $\times\varphi_n\left(i/n\right)Z_{in}$ is bounded by

$$\begin{aligned}
&\sup_{x\in[0,1]}\left|n^{-1}\sum\nolimits_{i=1}^{n}\Upsilon_n\left(i,x\right)\varepsilon_i - n^{-1}\sum\nolimits_{i=1}^{n}\Upsilon_n\left(i,x\right)Z_{in}\right| \\
&\leq \sup_{x\in[0,1]}\left|\Gamma_n(x)\right| + \sup_{x\in[0,1]}\left|n^{-1}\Upsilon_n\left(n,x\right)\left(S_n-W_{n,n}\right)\right|, \qquad \text{(S1.6)}
\end{aligned}$$

where $\Gamma_n(x) = n^{-1}\sum_{i=1}^{n-1}\left\{\Upsilon_n\left(i,x\right)-\Upsilon_n\left(i+1,x\right)\right\}\left(S_i-W_{i,n}\right)$. Assumption (E2) and $\varphi_n(x)\in \mathrm{Lip}\{[0,1], C_{\varphi,n}\}$ lead to

$$\sup_{1\leq i\leq n-1}\left|\sigma\left(i/n\right)\varphi_n\left(i/n\right)-\sigma\left(\left(i+1\right)/n\right)\varphi_n\left(\left(i+1\right)/n\right)\right| = \mathcal{O}\left(n^{-1}C_{\varphi,n}+\rho_n n^{-1}\right).$$

Therefore,

$$\begin{aligned}
&\sup_{1\leq i\leq n-1}\left|\Upsilon_n\left(i,x\right)-\Upsilon_n\left(i+1,x\right)\right| \\
&= \sup_{1\leq i\leq n-1}\Big|\tilde{K}_{\tilde{h}}\left(i/n-x\right)\sigma\left(i/n\right)\varphi_n\left(i/n\right) \\
&\qquad -\tilde{K}_{\tilde{h}}\left(\left(i+1\right)/n-x\right)\sigma\left(\left(i+1\right)/n\right)\varphi_n\left(\left(i+1\right)/n\right)\Big| \\
&\leq \sup_{1\leq i\leq n-1}\Big|\left\{\tilde{K}_{\tilde{h}}\left(i/n-x\right)-\tilde{K}_{\tilde{h}}\left(\left(i+1\right)/n-x\right)\right\}\sigma\left(i/n\right)\varphi_n\left(i/n\right) \\
&\qquad +U\left(n^{-1}\tilde{h}^{-1}C_{\varphi,n}+\rho_n\tilde{h}^{-1}n^{-1}\right)\Big| \\
&= U\left(\tilde{h}^{-2}n^{-1}\rho_n+n^{-1}\tilde{h}^{-1}C_{\varphi,n}\right). \qquad \text{(S1.7)}
\end{aligned}$$

According to Assumption (E6) and Equation (S1.7), for the first term of Equation (S1.6), one obtains that

$$\begin{aligned}
\sup_{x\in[0,1]}\left|\Gamma_n\left(x\right)\right| &\leq n^{-1}\left(\left[2n\tilde{h}\right]+1\right)\sup\nolimits_{1\leq i\leq n-1}\left|\Upsilon_n\left(i,x\right)-\Upsilon_n\left(i+1,x\right)\right| \\
&\times\max\nolimits_{1\leq i\leq n-1}\left|S_i-W_{i,n}\right| = \mathcal{O}_p\left(\tilde{h}^{-1}n^{\beta-1}\rho_n+n^{\beta-1}C_{\varphi,n}\right). \quad \text{(S1.8)}
\end{aligned}$$

For the second term of Equation (S1.6), according to Lemma A.3,

$$\sup_{x\in[0,1]}\left|n^{-1}\Upsilon_n\left(1,x\right)\left(S_n-W_{n,n}\right)\right| = \mathcal{O}_p\left(n^{\beta-1}\tilde{h}^{-1}\rho_n\right). \tag{S1.9}$$

Consequently, Equations (S1.6), (S1.8), (S1.9) and Lemma A.7 imply that

$$\begin{aligned}
&\sup_{x\in[0,1]}\left|n^{-1}\sum\nolimits_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n-x\right)\sigma\left(i/n\right)\varphi_n\left(i/n\right)\varepsilon_i\right|\\
&\leq \sup_{x\in[0,1]}\left|n^{-1}\sum\nolimits_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n-x\right)\sigma\left(i/n\right)\varphi_n\left(i/n\right)Z_{in}\right|\\
&\quad + U_p\left(\tilde{h}^{-1}n^{\beta-1}\rho_n+n^{\beta-1}C_{\varphi,n}\right)\\
&= U_p\left(n^{-1/2}\tilde{h}^{-1/2}\rho_n\log^{1/2}n+n^{\beta-1}C_{\varphi,n}+n^{\beta-1}\tilde{h}^{-1}\rho_n\right),
\end{aligned}$$

which yields Lemma A.8.

S2 Proof of Proposition 1

According to Assumption (M4), one has that $\left\|K^{(1)}(x)\right\|_\infty \leq C$. Then for $x\in\mathcal{I}_n$,

$$\begin{aligned}
\sup_{x\in\mathcal{I}_n}\left|\hat{f}(x)-1\right| &= \sup_{x\in\mathcal{I}_n}\left|n^{-1}\sum\nolimits_{i=1}^{n}K_h\left(i/n-x\right)-\int_0^1 K_h\left(u-x\right)du\right|\\
&= \sup_{x\in\mathcal{I}_n}\left|\sum\nolimits_{i=1}^{n}\int_{(i-1)/n}^{i/n}\left\{K_h\left(i/n-x\right)-K_h\left(u-x\right)\right\}du\right|\\
&\leq \sum\nolimits_{i=1}^{n}\int_{(i-1)/n}^{i/n}\sup_{x\in\mathcal{I}_n}\left|K_h\left(i/n-x\right)-K_h\left(u-x\right)\right|du\\
&= h^{-1}\sum\nolimits_{i=1}^{n}\int_{(i-1)/n}^{i/n}\sup_{x\in\mathcal{I}_n}\left|K\left(\frac{i/n-x}{h}\right)-K\left(\frac{u-x}{h}\right)\right|du\\
&\leq h^{-2}\sum\nolimits_{i=1}^{n}\int_{(i-1)/n}^{i/n}\left\|K^{(1)}(x)\right\|_\infty\left|u-i/n\right|du \leq Cn^{-1}h^{-2}.
\end{aligned}$$

This completes the proof of Proposition 1.

S3 Figures

This section shows some figures for the simulated data examples with the noise ε following a t-distribution and the strata pressure data examples.

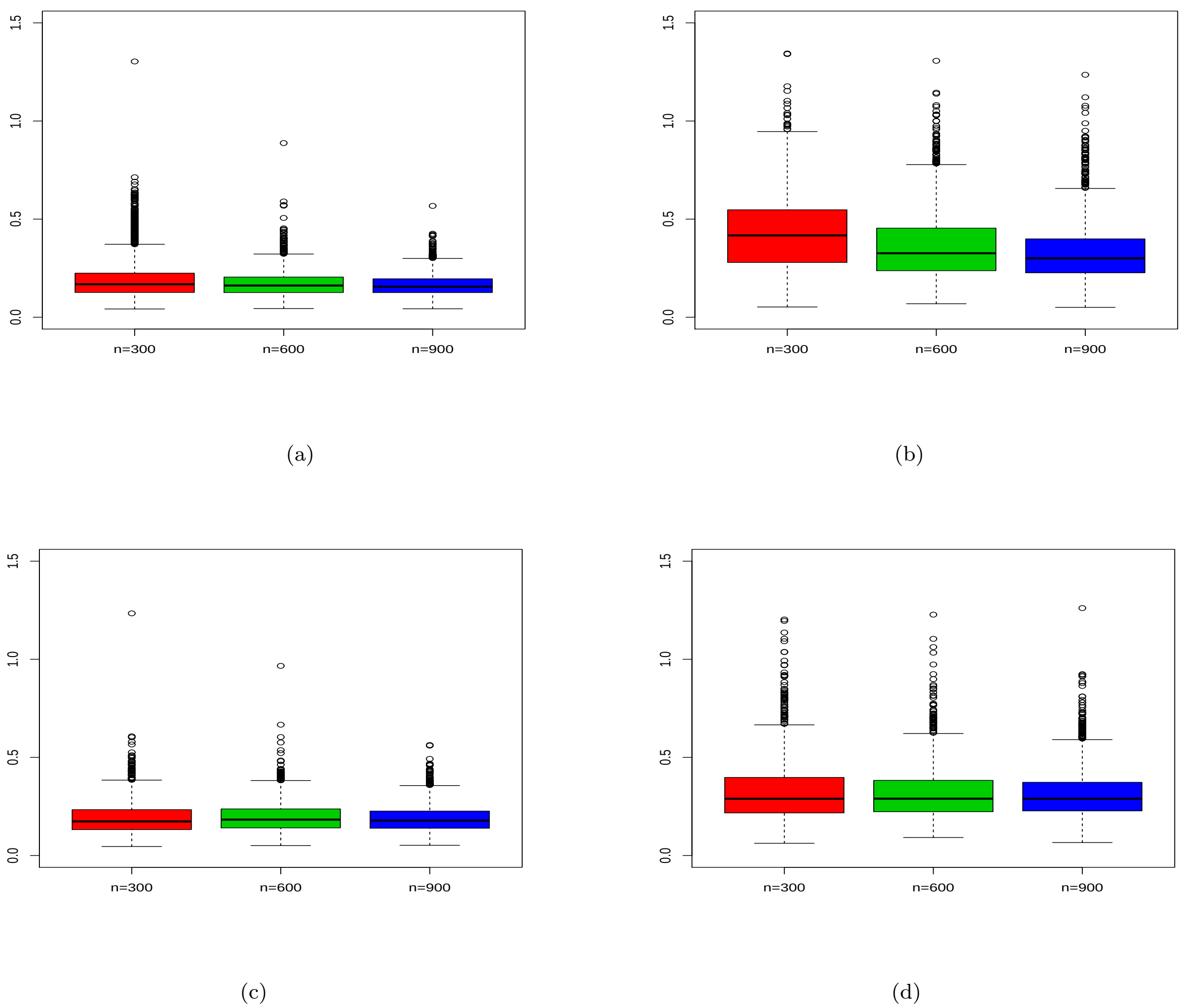
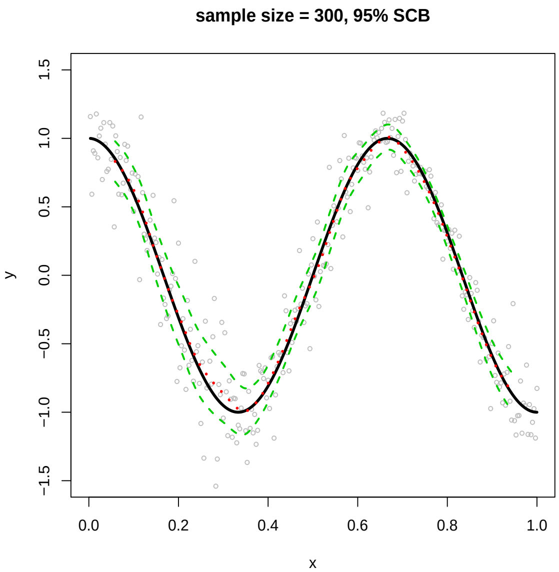
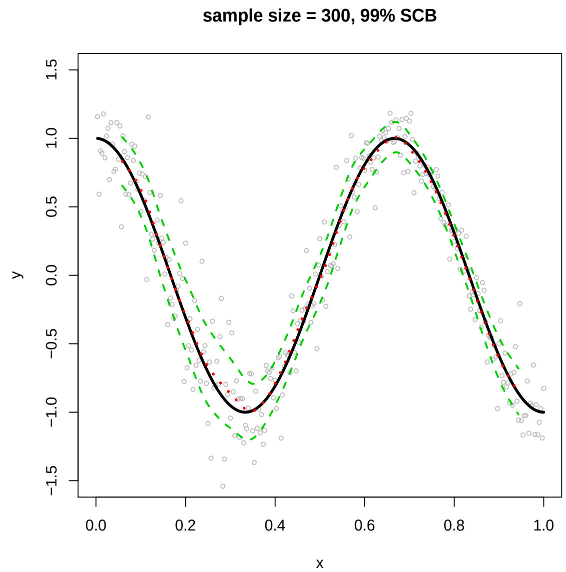


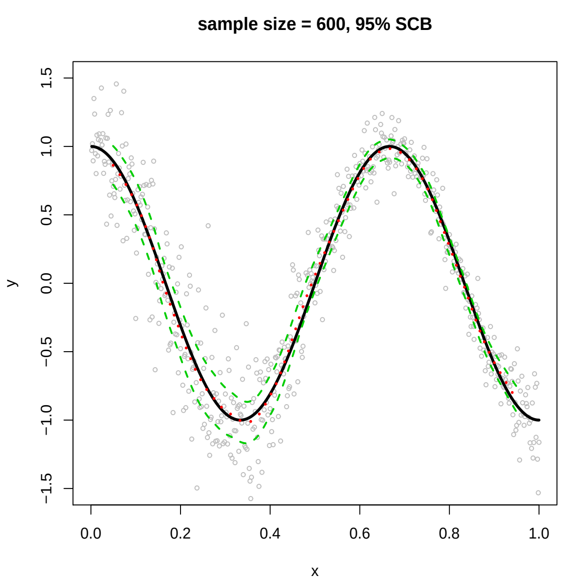
Figure S.1: Boxplots of $\triangle_n = \sqrt{n}\max_{j=1}^{400}\left|\tilde{\sigma}^2_{\mathrm{K}}\left(x_j\right) - \hat{\sigma}^2_{\mathrm{SK}}\left(x_j\right)\right|$ in which x_j are the equally spaced points on $\tilde{\mathcal{I}}_n$ over 2000 replications with $\varepsilon \sim \sqrt{0.8} * t_{10}$: (a) Case 1; (b) Case 2; (c) Case 3; (d) Case 4.



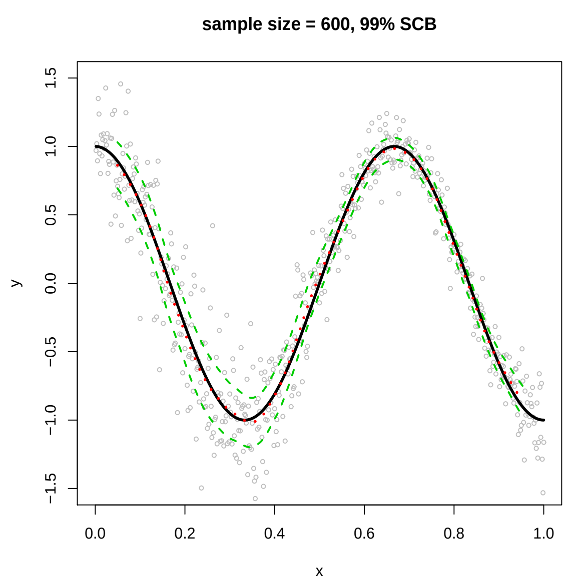
(a)



(b)

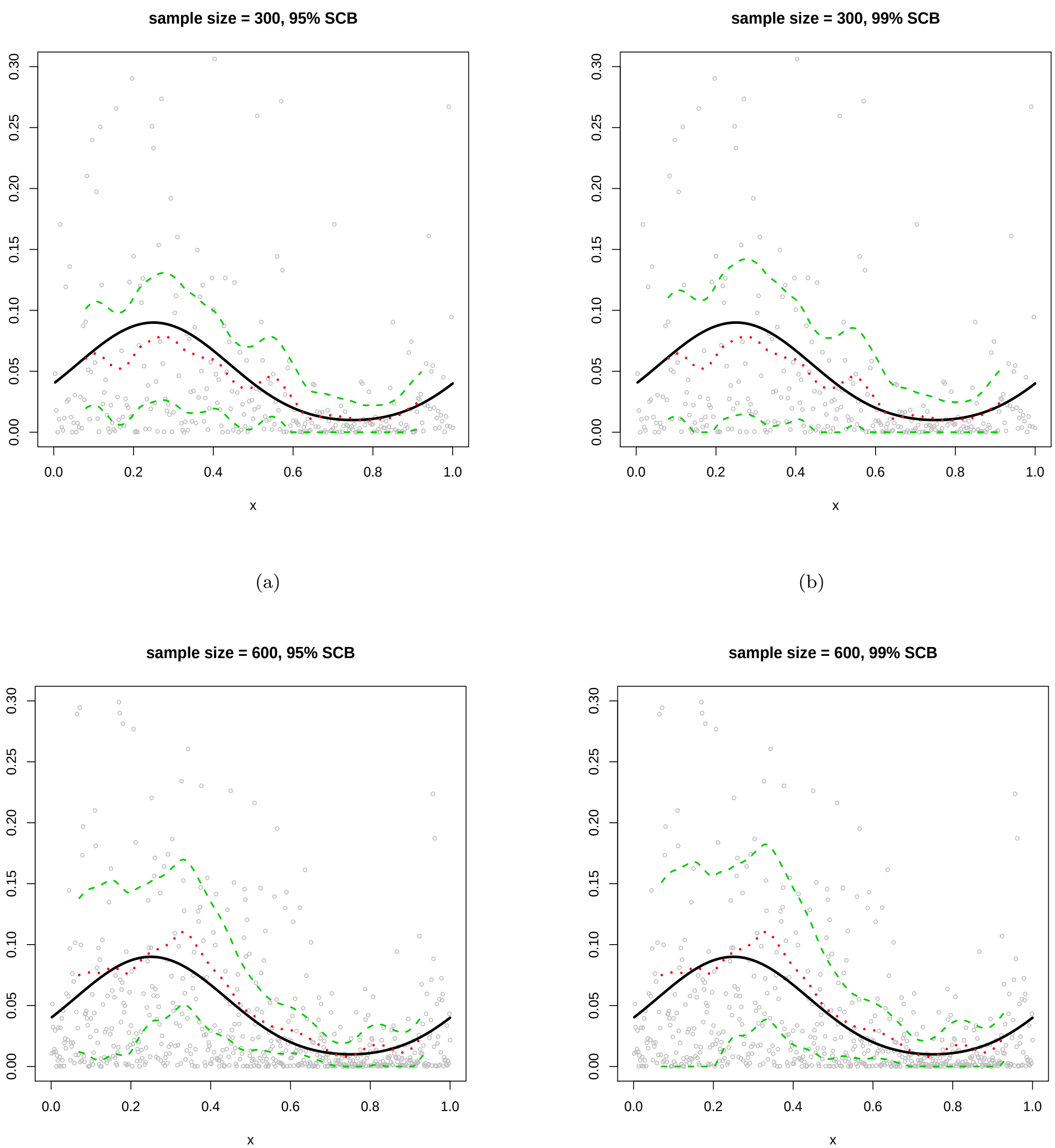


(c)



(d)

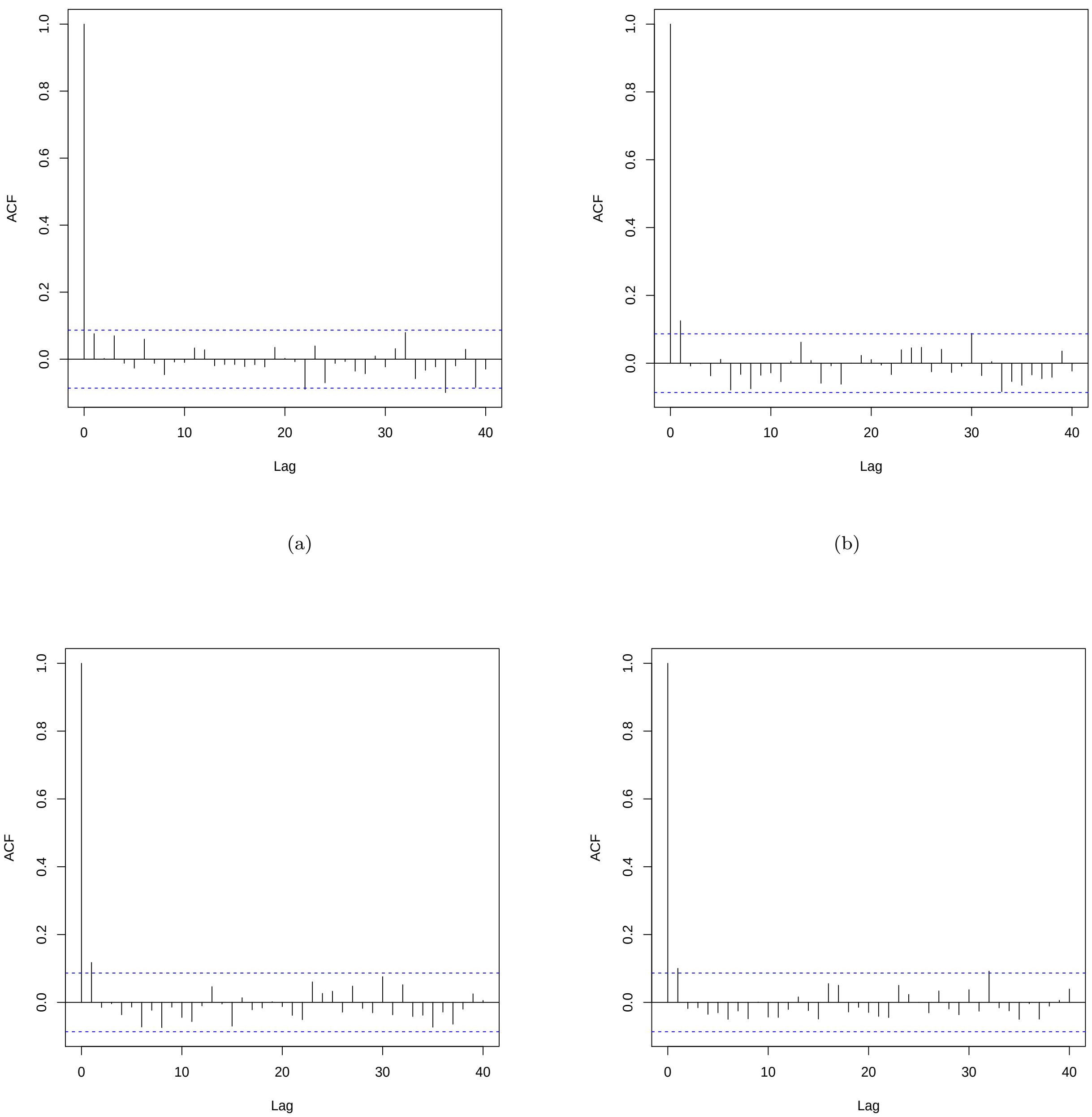
Figure S.2: Plots of SCB (dashed) for $m(x)$ (solid) in Case 1 with $\varepsilon \sim \sqrt{0.8} * t_{10}$ which is computed according to (4.2) and the estimator $\hat{m}(x)$ (dotted).



(a) (b)

(c) (d)

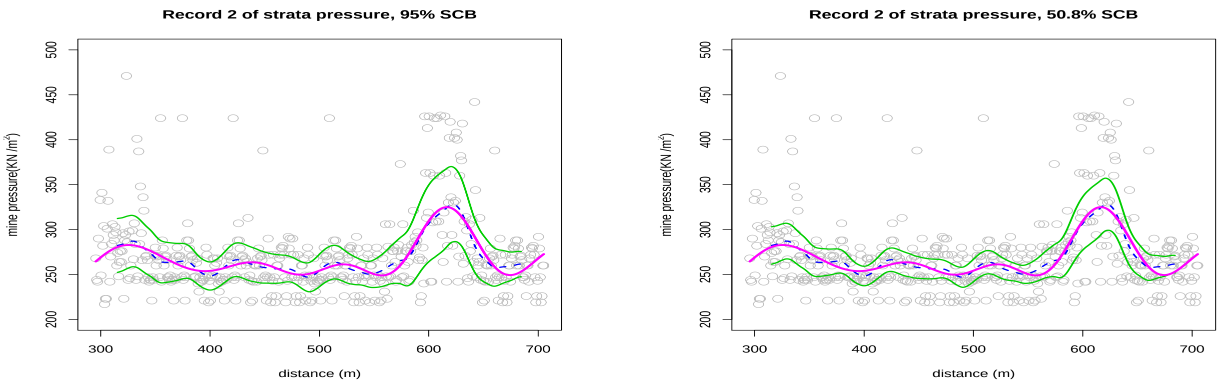
Figure S.3: Plots of SCB (dashed) for $\sigma^2(x)$ (solid) in Case 1 with $\varepsilon \sim \sqrt{0.8} * t_{10}$ and the estimator $\hat{\sigma}^2_{\mathrm{SK}}(x)$ (dotted).



(a) (b)

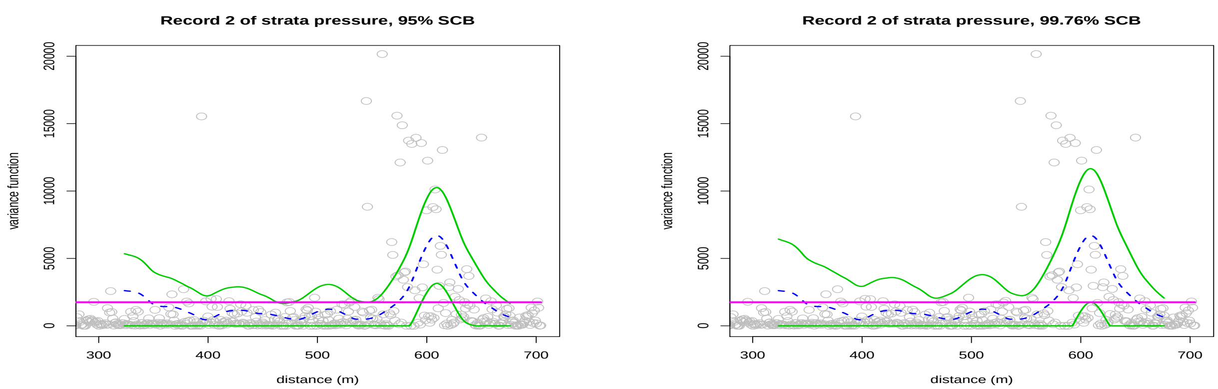
(c) (d)

Figure S.4: For record 1, plots of the acfs of (a) $\{\hat{\varepsilon}_i\}_{i=1}^n$, (b) $\{|\hat{\varepsilon}_i|\}_{i=1}^n$, (c) $\{\hat{\varepsilon}_i^2\}_{i=1}^n$, (d) $\{\hat{\varepsilon}_i^4\}_{i=1}^n$.



(a) (b)

Figure S.5: For record 2, plots of the null hypothesis curve of $m(x) = a_0 + \sum_{k=1}^{5}\{a_k \sin(k\omega x) + b_k \cos(k\omega x)\}$ (thick solid), kernel estimator $\hat{m}(x)$ (dashed), SCB (solid) for $m(x)$ with (a) $\alpha = 0.05$ and (b) $\alpha = 0.492$.



(a) (b)

Figure S.6: For record 2, plots of the null hypothesis curve of $\hat{\sigma}^2 = n^{-1}\sum_{i=1}^{n}\hat{e}_i^2$ (thick solid), SCB (solid) for $\sigma^2(x)$ and the spline-kernel estimator $\hat{\sigma}^2_{\mathrm{SK}}(x)$ (dashed) with (a) $\alpha = 0.05$ and (b) 0.0024.