SIMULTANEOUS CONFIDENCE BANDS FOR MEAN AND VARIANCE FUNCTIONS BASED ON DETERMINISTIC DESIGN

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Abstract: Asymptotically correct simultaneous confidence bands (SCBs) are proposed for the mean and variance functions of nonparametric regression model based on deterministic designs. The variance estimation is as efficient up to order $n^{-1/2}$ as an infeasible estimator if the mean function were known. Simulation experiments provide strong evidence that corroborates the asymptotic theory. The proposed SCBs are used to analyze two sets of strata pressure data from the Bullianta Coal Mine in Erdos City, Inner Mongolia, China.

Key words and phrases: Brownian motion, heteroscedasticity, kernel, oracle efficiency, spline, strata pressure.

1. Introduction

Simultaneous confidence intervals (SCIs) have long been recognized as vital tools for inference on the global shape of curves; see, for instance, Stapleton (2009) Section 5.2 for the Scheffé SCIs of simple linear regression function, and Section 5.3 for Tukey SCIs of surface of contrasts. In the more complicated context of nonparametric function estimation, SCIs are generally known as simultaneous confidence bands (SCBs), and were first constructed in Bickel and Rosenblatt (1973) for probability density function, and extended by Johnston (1982) and Härdle (1989) to univariate kernel regression. Xia (1998) proposed bias-corrected SCBs based on local polynomial fitting under assumption of homoscedasticity, while Härdle and Bowman (1988), Härdle and Marron (1991), Keilegom and Claeskens (2003) and Neumann and Polzehl (1998) studied bootstrap kernel SCBs.

More recently, nonparametric SCB methodology has diversified in both techniques and scope. For instance, Wang and Yang (2009) proposed SCBs for nonparametric regression function based on polynomial spline, which was extended in Song and Yang (2009) to oracally efficient spline SCBs for conditional variance function. Cai and Yang (2015) improved over Song and Yang (2009) by a spline-kernel oracally efficient two step estimator for the variance function with SCBs. Wang, Cheng and Yang (2013) proposed smooth SCBs for cumulative distribution functions, which was shown in Wang et al. (2014) to be oracally efficient for the cumulative distribution function of unobserved autoregressive errors. Degras (2011), Cao et al. (2012), Ma et al. (2012), Zheng, Yang and Härdle (2014), Gu et al. (2014) and Cao et al. (2016) constructed various SCBs for the mean and covariance functions of functional data, while Gu and Yang (2015) established oracle efficiency of an SCB for the single-index link function.

Existing literature on SCBs for nonparametric regression concerns mostly the random design model $Y_i = m(X_i) + \sigma(X_i)\varepsilon_i$ with observed i.i.d. points $\{(X_i, Y_i)\}_{i=1}^n$ and errors $\{\varepsilon_i\}_{i=1}^n$. Often encountered in applications (e.g., strata pressure data discussed in Subsection 5.2) is the following deterministic design nonparametric regression model

$$Y_i = m\,(i/n) + \sigma(i/n)\varepsilon_i \tag{1.1}$$

in which the Y_i's are responses at equally spaced design points $i/n, 1 \le i \le n$, and $\{\varepsilon_i\}_{i=1}^n$ are unobserved i.i.d. random errors with $\mathrm{E}\,(\varepsilon_1) = 0$, $\mathrm{var}\,(\varepsilon_1) = 1$. Assume that there are smooth but unknown mean and variance functions $m(x)$ and $\sigma^2(x)$ that satisfy model (1.1) for all n. In this paper, we aim to construct asymptotically correct SCBs for both the mean function $m(x)$ and variance function $\sigma^2(x)$ in model (1.1) without restrictive assumptions. As an illustration, the SCBs for the mean and variance functions are applied to two strata pressure data sets collected from the Bulianta Coal Mine located in Ordos City, Inner Mongolia, China. Figures 4 and 5 depict the SCBs for one set of the data, and the SCBs for the second data set are given in Figures S.5 and S.6 in the online supplement. For both data sets, the null hypothesis of mean function being $m(x) = a_0 + \sum_{k=1}^5 \{a_k \sin(k\omega x) + b_k \cos(k\omega x)\}$ for some constants a_0, a_k and b_k $(k = 1, \ldots, 5)$ cannot be rejected with the p-values as high as 0.847 and 0.545 respectively. Meanwhile, the SCB for variance function is used to test the homoscedasticity null hypothesis for the two real data sets. The conclusions are (i) strong rejection for one with the p-value $= 0.0024$ and (ii) no rejection for the other with the p-value $= 0.545$; see Subsection 5.2 for details.

Based on design model (1.1), Donoho and Johnstone (1996) studied adaptive nonparametric estimation for the mean function, and Angelini et al. (2003) proposed a nonparametric estimator of the mean function and showed that the estimator is the best linear unbiased predictor. SCBs for the mean function in model (1.1) were studied in Hall and Titterington (1988) and more recently in Cai, Low and Ma (2014). These SCBs are adaptive for $m(x)$ belonging to some function class, but as a result asymptotically conservative instead of asymptotically correct. A more serious limitation of these adaptive SCBs is their reliance on assumptions that the $\{\varepsilon_i\}_{i=1}^n$ are i.i.d. $N(0,1)$ and the variance function $\sigma^2(x)$ is constant. Alternatively, Eubank and Speckman (1993) obtained SCB for the mean function $m(x)$ based on kernel smoothing, but under the restrictive assumption of homoscedasticity ($\sigma^2(x) \equiv \sigma^2$) and the mean function $m(x)$ being periodic. None of these previous works, therefore, can handle clearly heteroscedastic data, such as the strata pressure data. More recently, Wang (2012) constructed a spline SCB for the mean function $m(x)$ based on deterministic designs and $\{\varepsilon_i\}_{i=1}^n$ being α-mixing, but its asymptotically conservative coverage limits its usefulness for testing hypotheses. For variance function estimation, Brown and Levine (2007) and Levine (2006) proposed difference-based kernel estimators and an approach of bandwidth selection respectively and later Cai, Levine and Wang (2009) extended them to the multivariate situation and established the minimax convergence rate in the i.i.d. Gaussian case. Meanwhile, Wang et al. (2008) studied the effect of the unknown mean on the variance function estimation function in nonparametric regression. However, there are no SCBs for the variance function in the works above.

The rest of the paper is organized as follows. Section 2 establishes the main asymptotic theoretical results. Section 3 provides insights of proofs and Section 4 gives concrete steps to implement the SCBs. Section 5 reports some simulation results and real data analyses. The technical proofs are given in the Appendix and supplementary material.

2. Main Results

2.1 SCB for the mean function

We first formulate an SCB for the mean function $m(x)$ in model (1.1) by smoothing the data set $\{(i/n, Y_i)\}_{i=1}^n$ to approximate $m(x)$. The basic idea is to find a locally weighted least squares estimate $\hat{m}(x)$ which solves the following minimization problem:

$$\min_\theta n^{-1} \sum_{i=1}^n (Y_i - \theta)^2 K_h(i/n - x) = n^{-1} \sum_{i=1}^n \{Y_i - \hat{m}(x)\}^2 K_h(i/n - x),$$

in which $K(u)$ is a kernel function, $h = h_n > 0$ is a sequence of smoothing parameters called bandwidth, $K_h(u) = h^{-1} K(u/h)$ is the kernel function rescaled by h. Clearly,

$$\hat{m}(x) = \frac{n^{-1} \sum_{i=1}^n K_h(i/n - x) Y_i}{\hat{f}(x)}, \tag{2.1}$$

where $\hat{f}(x) = n^{-1} \sum_{i=1}^n K_h(i/n - x)$. The estimator $\hat{m}(x)$ was proposed by Nadaraya (1964) and Watson (1964) and commonly referred to as the Nadaraya-Watson estimator.

In the following, we denote by $\psi^{(s)}(x)$ the s-th order derivative of a function $\psi(x)$. For $\theta \in (0,1]$ and integer $p \geq 0$, let $C^{p,\theta}[0,1]$ be the space of functions with $\theta-$Hölder continuous p-th-order derivatives on $[0,1]$,

$$C^{p,\theta}[0,1] = \left\{ \phi(x) : \|\phi\|_{p,\theta} = \sup_{x \neq x', x, x' \in [0,1]} \frac{\left|\phi^{(p)}(x) - \phi^{(p)}(x')\right|}{|x - x'|^\theta} < +\infty \right\},$$

and denote by $C^{(p)}[0,1]$ the space of p-times continuously differentiable functions. For sequences of positive real numbers c_n and d_n, $c_n \ll d_n$ means $c_n/d_n \to 0$ as $n \to \infty$.

We require the following assumptions to construct SCBs for $m(x)$.

(M1) *The function* $m(\cdot) \in C^{p-1,\theta}[0,1]$ *for integer* $p > 1$ *and* $\theta \in (0,1]$.

(M2) *The error* ε *satisfies* $\mathrm{E}(\varepsilon) = 0$, $\mathrm{E}(\varepsilon^2) = 1$ *and* $\sigma^2(x) \in C^{(1)}[0,1]$ *with* $0 < c_\sigma \leq \sigma^2(x) \leq C_\sigma < +\infty$ *for any* $x \in [0,1]$.

(M3) *There exist* $\beta \in (0, 1/2 - 1/(4\theta + 4p - 2))$, $C_0 \in (0, +\infty)$, $\gamma \in (1, +\infty)$, *and i.i.d.* $N(0,1)$ *variables* $\{Z_{in}\}_{i=1}^n$ *such that*

$$P\left\{ \max_{1 \leq l \leq n} \left| \sum\nolimits_{i=1}^l \varepsilon_i - \sum\nolimits_{i=1}^l Z_{in} \right| > n^\beta \right\} < C_0 n^{-\gamma}.$$

(M4) *The kernel function* $K \in C^{(1)}(\mathbb{R})$, *is of order* p, *and is supported on* $[-1,1]$.

(M5) *The bandwidth* $h = h_n$ *satisfies* $\log h_n / (-\log n) \to t > 0$ *as* $n \to \infty$ *and*

$$\max\left\{n^{-1/2}\log^{1/2} n, n^{2\beta-1}\log n\right\} \ll h_n \ll (n\log n)^{-1/(2\theta+2p-1)}.$$

Hence $1/(2\theta+2p-1) \le t \le \min\{1/2, 1-2\beta\}$.

Assumptions (M1), (M2) and (M4) are typical for kernel smoothing which are adapted from Härdle (1989) and Eubank and Speckman (1993). Assumption (M5) is the general condition on the choice of bandwidth h. It is more convenient to make the inequalities on t strict in (M5). The same holds true for $\tilde{t}$ and τ in Assumptions (E5) and (E7) below. Assumption (M3) provides the Gaussian approximation of the error process. According to Lemma A.2 in the Appendix, Assumption (M3) is ensured by the following elementary Assumption (M3'):

(M3') *There exists* $\eta > 2/\beta - 2$, $\beta \in (0, 1/2 - 1/(4\theta+4p-2))$ *such that* $\mathrm{E}|\varepsilon_1|^{2+\eta} < +\infty$.

Let $\mathcal{I}_n = [h_n, 1-h_n]$. The SCB for $m(x)$ in the following theorem is a direct corollary of Propositions 1–4 in Section 3.

Theorem 1. *Under Assumptions (M1)–(M5), as* $n \to \infty$,

$$P\left\{a_h[\sup_{x\in\mathcal{I}_n} |\{m(x) - \hat{m}(x)\}/v(x)| - b_h] \le z\right\} \to \exp\{-2\exp(-z)\}, z \in \mathbb{R},$$

where $a_h = \left\{2\log h^{-1}\right\}^{1/2}$, $b_h = a_h + a_h^{-1}\left\{2^{-1}\log\left(C_K/\left(4\pi^2\right)\right)\right\}$,

$$C_K = \int_{-1}^{1} K^{(1)}(v)^2\, dv \Big/ \int_{-1}^{1} K(v)^2\, dv, v(x) = (nh)^{-1/2}\sigma(x)\left(\int_{-1}^{1} K^2(u)\, du\right)^{1/2}.$$

That is, for any $\alpha \in (0,1)$,

$$P\left\{m(x) \in \hat{m}(x) \pm v(x)\left[a_h + a_h^{-1}\left\{q_\alpha + 1/2\log\left(C_K/\left(4\pi^2\right)\right)\right\}\right], \forall x \in \mathcal{I}_n\right\} \to 1-\alpha,$$

in which $q_\alpha = -\log\{-1/2\log(1-\alpha)\}$.

Theorem 1 implies that the SCB contracts to zero at the rate $n^{-1/2}h^{-1/2}\log^{1/2} n$. In the special case of $p = 2, \theta = 1$ as in Subsection 4.1, the implemented order of h satisfying assumption (M5) is $n^{-1/5}\log^{-1/5-\delta_1} n$ for any $\delta_1 > 0$. That is, the optimal bandwidth order of $n^{-1/5}$ is under smoothed by $\log^{-1/5-\delta_1} n$, and the contraction rate of SCB is $n^{-2/5}\log^{3/5+\delta_1/2} n$.

2.2 SCBs for the variance function

The variance function $\sigma^2(x)$ measures the heteroscedastic variation of the errors $e_i = Y_i - m(i/n), 1 \le i \le n$ in model (1.1). Following Cai and Yang (2015), if $m(x)$ were known by 'oracle', one could compute the squared errors $\{e_i^2\}_{i=1}^n$, and then by smoothing the data $\{(i/n, e_i^2)\}_{i=1}^n$ obtain a would-be kernel estimator of $\sigma^2(x)$:

$$\tilde{\sigma}_{\mathrm{K}}^2(x) = \frac{n^{-1}\sum_{i=1}^n \tilde{K}_{\tilde{h}}(i/n - x)\, e_i^2}{n^{-1}\sum_{i=1}^n \tilde{K}_{\tilde{h}}(i/n - x)},$$

where $\tilde{K}(u)$ is a kernel function and $\tilde{h} = \tilde{h}_n > 0$ a bandwidth. However, $\tilde{\sigma}_{\mathrm{K}}^2(x)$ is infeasible as the errors $\{e_i^2\}_{i=1}^n$ are unobservable. To mimic $\tilde{\sigma}_{\mathrm{K}}^2(x)$, a spline-kernel estimator $\hat{\sigma}_{\mathrm{SK}}^2(x)$ is proposed

$$\hat{\sigma}_{\mathrm{SK}}^2(x) = \frac{n^{-1}\sum_{i=1}^n \tilde{K}_{\tilde{h}}(i/n - x)\, \hat{e}_i^2}{n^{-1}\sum_{i=1}^n \tilde{K}_{\tilde{h}}(i/n - x)}, \tag{2.2}$$

where $\hat{e}_i = Y_i - \hat{m}_p(i/n)$ and $\hat{m}_p(x)$ is the p-th order spline estimator for $m(x)$ with integer $p > 0$,

$$\hat{m}_p(x) = \arg\min_{g \in \mathcal{H}_N^{(p-2)}} \sum_{i=1}^n \{Y_i - g(i/n)\}^2, \tag{2.3}$$

in which $\mathcal{H}_N^{(p-2)} = \mathcal{H}_N^{(p-2)}[0,1]$ is the space of spline functions on interval $[0,1]$ defined below.

Divide the interval $[0,1]$ into $(N+1)$ subintervals $J_j = [\chi_j, \chi_{j+1}),\ j = 0, 1, 2, \ldots, N$ by equally spaced points $\{\chi_j\}_{j=1}^N$ called interior knots,

$$\chi_0 = 0 < \cdots < 1 = \chi_{N+1},\ \chi_j = j/(N+1),\ j = 0, 1, \ldots, N+1.$$

$\mathcal{H}_N^{(p-2)}$ is the space of functions that are polynomials of degree $(p-1)$ on each J_j with continuous $(p-2)$-th derivative on $[0,1]$. For instance, $\mathcal{H}_N^{(-1)}$ consists of functions that are constant on each J_j, and $\mathcal{H}_N^{(0)}$ the space of functions that are linear on each J_j and continuous on $[0,1]$.

Let $\mathrm{E}\left(\varepsilon_i^4\right) = \mu_4$ and $\eta_i = \left(\varepsilon_i^2 - 1\right)(\mu_4 - 1)^{-1/2}$. We require the following Assumptions (E1)–(E7) to construct SCBs for $\sigma^2(x)$:

(E1) *The function* $m(\cdot) \in C^p[0,1]$ *for integer* $p > 1$.

(E2) *The error* ε *satisfies* $\mathrm{E}(\varepsilon) = 0$, $\mathrm{E}(\varepsilon^2) = 1$ *and* $\sigma^2(x) \in C^{p_0-1,\theta_0}[0,1]$ *for integer* $p_0 > 1$, $\theta_0 \in (0,1]$ *with* $0 < c_\sigma \leq \sigma^2(x) \leq C_\sigma < +\infty$ *for any* $x \in [0,1]$.

(E3) *There exist* $\beta' \in (0, 1/2 - 1/(4\theta_0 + 4p_0 - 2))$, $C_0' \in (0,+\infty)$, $\gamma' \in (1,+\infty)$, *and i.i.d.* $N(0,1)$ *variables* $\{Z_{in}'\}_{i=1}^n$ *such that*

$$P\left\{\max_{1\leq l\leq n}\left|\sum\nolimits_{i=1}^{l}\eta_i - \sum\nolimits_{i=1}^{l} Z_{in}'\right| > n^{\beta'}\right\} < C_0' n^{-\gamma'}.$$

(E4) *The kernel function* $\tilde{K} \in C^{(1)}(\mathbb{R})$, *is of order* p_0, *and is supported on* $[-1,1]$.

(E5) *The bandwidth* $\tilde{h} = \tilde{h}_n$ *satisfies* $\log\tilde{h}_n/(-\log n) \to \tilde{t} > 0$ *as* $n \to \infty$ *and*

$$\max\left\{n^{-1/2}\log^{1/2} n, n^{2\beta'-1}\log n, n^{-2(p-1)/(2p+1)}\right\} \ll \tilde{h} \ll (n\log n)^{-1/(2\theta_0+2p_0-1)}.$$

Consequently, $1/(2\theta_0 + 2p_0 - 1) \leq \tilde{t} \leq \min\left\{1/2, 1-2\beta', 2(p-1)/(2p+1)\right\}$.

(E6) *There exist* $C_0 \in (0,+\infty)$, $\gamma \in (1,+\infty)$, $\beta \in (0,b]$ *and i.i.d.* $N(0,1)$ *variables* $\{Z_{in}\}_{i=1}^n$ *such that*

$$P\left\{\max_{1\leq l\leq n}\left|\sum\nolimits_{i=1}^{l}\varepsilon_i - \sum\nolimits_{i=1}^{l} Z_{in}\right| > n^{\beta}\right\} < C_0 n^{-\gamma},$$

where $b = \min\{1 - 3/2(2p+1) - \tilde{t}, 1 - 5/2(2p+1) - 5\tilde{t}/2(2p+3), 1/2 - 1/(4\theta_0 + 4p_0 - 2)\}$.

(E7) *The number of interior knots* N *satisfies* $\log N/\log n \to \tau$ for some $\tau > 0$ *and*

$$\max\left\{n^{1/4p}, \tilde{h}^{-1/(p-1)}n^{(\beta-1/2)/(p-1)}, \tilde{h}^{-1/2(p-1)}\log^{1/2(p-1)} n\right\} \ll N \ll \min\left\{\tilde{h}^{2/3}n^{2(1-\beta)/3}, n^{2(1-\beta)/5}, n^{1/3}\tilde{h}^{1/3}\log^{-1/3} n\right\}.$$

Consequently, $\max\{1/4p, \left(2\tilde{t} + 2\beta - 1\right)/2(p-1), \tilde{t}/2(p-1)\} \leq \tau \leq$

$$\min\{2(1-\beta)/3 - 2\tilde{t}/3, 2(1-\beta)/5, 1/3 - \tilde{t}/3\}.$$

Assumptions (E2)–(E5) are adapted from Assumptions (M1)–(M5) of Subsection 2.1. Assumption (E1) is a general condition for spline regression of mean function in model (1.1), while Assumption (E7) on the choice of knots number N ensures the oracle efficiency and the extreme distribution result in the following (2.4). Lemma A.2 in the Appendix imply that Assumptions (E3) and (E6) are ensured by the following Assumption (E3').

(E3') *There exists* $\eta' > 2/\beta - 2$, $\beta \in (0, b]$ *as in Assumption (E6) such that* $\mathrm{E}|\varepsilon_1|^{4+2\eta'} < +\infty$.

Under Assumptions (E2)–(E5), applying Theorem 1 to unobservable sample $\{(i/n, e_i^2)\}_{i=1}^n$ and letting $\tilde{\mathcal{I}}_n = \left[\tilde{h}_n, 1-\tilde{h}_n\right]$, for each $z \in \mathbb{R}$ one has

$$P\left\{a_{\tilde{h}}\left(\sup_{x\in\tilde{\mathcal{I}}_n}\left|\left\{\sigma^2(x) - \tilde{\sigma}_{\mathrm{K}}^2(x)\right\}/v_0(x)\right| - b_{\tilde{h}}\right) \le z\right\} \to \exp\{-2\exp(-z)\}, \tag{2.4}$$

where

$$a_{\tilde{h}} = \left\{2\log\tilde{h}^{-1}\right\}^{1/2}, \; b_{\tilde{h}} = a_{\tilde{h}} + a_{\tilde{h}}^{-1}\left\{2^{-1}\log\left(C_{\tilde{K}}/\left(4\pi^2\right)\right)\right\}, \tag{2.5}$$

$$C_{\tilde{K}} = \int_{-1}^{1}\tilde{K}^{(1)}(v)^2\,dv \Big/ \int_{-1}^{1}\tilde{K}(v)^2\,dv, \tag{2.6}$$

$$v_0(x) = \left\{n^{-1}\tilde{h}^{-1}\sigma_0^2(x)\int_{-1}^{1}\tilde{K}^2(u)\,du\right\}^{1/2}, \tag{2.7}$$

in which $\sigma_0^2(x) = \sigma^4(x)(\mu_4 - 1)$ so that $\mathrm{var}(e_i^2) = \mathrm{E}\left(e_i^4\right) - \left\{\mathrm{E}\left(e_i^2\right)\right\}^2 = \sigma_0^2(i/n)$, as the second and fourth moments of e_i are $\mathrm{E}\left(e_i^2\right) = \sigma^2(i/n), \mathrm{E}\left(e_i^4\right) = \sigma^4(i/n)\mu_4$.

According to (2.4), it is obvious that an asymptotic $100(1-\alpha)\%$ 'infeasible' SCB for $\sigma^2(x)$ over $\tilde{\mathcal{I}}_n = \left[\tilde{h}_n, 1-\tilde{h}_n\right]$ is

$$\tilde{\sigma}_{\mathrm{K}}^2(x) \pm v_0(x)\left[a_{\tilde{h}} + a_{\tilde{h}}^{-1}\left\{q_\alpha + 1/2\log\left(C_{\tilde{K}}/\left(4\pi^2\right)\right)\right\}\right]. \tag{2.8}$$

Theorem 2. *Under Assumptions (E1)–(E7), as* $n \to \infty$, *the spline-kernel estimator* $\hat{\sigma}_{SK}^2(x)$ *is oracally efficient, i.e., it is asymptotically as efficient as the 'infeasible' estimator* $\tilde{\sigma}_K^2(x)$*:*

$$\sup_{x\in[0,1]}\left|\hat{\sigma}_{SK}^2(x) - \tilde{\sigma}_K^2(x)\right| = o_p\left(n^{-1/2}\right).$$

The proof of Theorem 2 depends on Propositions 5–7 given in Subsection 3.2. Theorem 1, Theorem 2 and Slutsky's Theorem together imply the following result:

Theorem 3. *Under Assumptions (E1)–(E7), as* $n \to \infty$, *an asymptotic* 100 $(1-\alpha)\%$ *SCB for* $\sigma^2(x)$ *over* $\tilde{\mathcal{I}}_n = \left[\tilde{h}_n, 1-\tilde{h}_n\right]$ *is*

$$\hat{\sigma}^2_{SK}(x) \pm v_0(x)\left[a_{\tilde{h}} + a_{\tilde{h}}^{-1}\left[q_\alpha + 1/2\log\left\{C_{\tilde{K}}/\left(4\pi^2\right)\right\}\right]\right]$$

with $a_{\tilde{h}}$, $C_{\tilde{K}}$, $v_0(x)$ *and* q_α *given in (2.5), (2.6), (2.7) and Theorem 1 respectively.*

Theorem 3 implies that the SCB contracts to zero at the rate $n^{-1/2}\tilde{h}^{-1/2}\log^{1/2} n$. In the special case of $p=4, p_0=2, \theta_0=1$ as in Subsection 4.2, the implemented order of $\tilde{h}$ satisfying assumption (E5) is $n^{-1/5}\log^{-1/5-\delta_2}$ for any $\delta_2 > 0$. That is, the optimal bandwidth order of $n^{-1/5}$ is under smoothed by $\log^{-1/5-\delta_2} n$, and the contraction rate of the SCB is $n^{-2/5}\log^{3/5+\delta_2/2} n$.

3. Error Decomposition

3.1 Case of the mean function

An asymptotic SCB for $m(x)$ is constructed by examining $\sup_{x\in\mathcal{I}_n}|\hat{m}(x) - m(x)|$. Note that

$$\begin{aligned}
\hat{m}(x) - m(x) &= n^{-1}\hat{f}(x)^{-1}\sum_{i=1}^{n} K_h\left(i/n - x\right) Y_i - m(x) \\
&= n^{-1}\hat{f}(x)^{-1}\sum_{i=1}^{n} K_h\left(i/n - x\right)\left\{m\left(i/n\right) - m(x) + \sigma(i/n)\varepsilon_i\right\} \\
&= \hat{f}(x)^{-1}\left\{A_n(x) + B_n\left(x\right)\right\},
\end{aligned}$$

in which

$$A_n(x) = n^{-1}\sum_{i=1}^{n} K_h\left(i/n - x\right)\left\{m\left(i/n\right) - m(x)\right\}, \tag{3.1}$$

$$B_n(x) = n^{-1}\sum_{i=1}^{n} K_h\left(i/n - x\right)\sigma(i/n)\varepsilon_i. \tag{3.2}$$

The following stochastic processes approximate $B_n(x)$:

$$B_{n1}(x) = n^{-1}\sum_{i=1}^{n} K_h\left(i/n - x\right)\sigma(i/n)Z_{in}, \tag{3.3}$$

$$B_{n2}(x) = n^{-1}\sum_{i=1}^{n} K_h\left(i/n - x\right)\sigma(x)Z_{in}, \tag{3.4}$$

$$B_{n3}(x) = n^{-1/2}\int K_h\left(u - x\right)\sigma\left(x\right)dW_n(u), \tag{3.5}$$

where $\{Z_{in}\}_{i=1}^{n}$ are i.i.d. $N\left(0,1\right)$ variables satisfying (M3) and $W_n\left(u\right)$ a two-sided Brownian motion on $(-\infty,+\infty)$ satisfying $Z_{in}=\sqrt{n}\left\{W_n\left(i/n\right)-W_n\left(\left(i-1\right)/n\right)\right\}$.

Proposition 1. *Under Assumption (M4), as $n\to\infty$,*

$$\sup_{x\in\mathcal{I}_n}\left|\hat{f}(x)-1\right| = \mathcal{O}\left(n^{-1}h^{-2}\right).$$

Proposition 2. *Under Assumptions (M1), (M4) and (M5), as $n\to\infty$,*

$$\sup_{x\in\mathcal{I}_n}\left|A_n(x)\right| = \mathcal{O}\left(h^{\theta+p-1}+n^{-1}h^{-1}\right).$$

Proposition 3. *Under Assumptions (M2)–(M4), as $n\to\infty$,*

$$\begin{aligned}
&(a) \quad \sup\nolimits_{x\in[0,1]}\left|B_n(x)-B_{n1}(x)\right| = \mathcal{O}_p\left(n^{\beta-1}h^{-1}\right),\\
&(b) \quad \sup\nolimits_{x\in[0,1]}\left|B_{n1}(x)-B_{n2}(x)\right| = \mathcal{O}_p\left(n^{-1/2}h^{1/2}\log^{1/2}n\right),\\
&(c) \quad \sup\nolimits_{x\in\mathcal{I}_n}\left|B_{n2}(x)-B_{n3}(x)\right| = \mathcal{O}_p\left(n^{-3/2}h^{-2}\log^{1/2}n\right),\\
&(d) \quad \sup\nolimits_{x\in[0,1]}\left|B_{n3}(x)\right| = \mathcal{O}_p\left(n^{-1/2}h^{-1/2}\log^{1/2}n\right).
\end{aligned}$$

The proofs of the propositions above are given in the Appendix.

Notice that $\mathrm{E}\{B_{n3}^2(x)\} = n^{-1}h^{-1}\sigma^2\left(x\right)\int_{-1}^{1}K^2\left(u\right)du$. Standardizing the process $B_{n3}(x)$ for $x\in[0,1]$, one obtains the following standard Gaussian process

$$\int K_h\left(x-u\right)dW_n\left(u\right)/\left\{h^{-1}\int_{-1}^{1}K^2\left(u\right)du\right\}^{1/2}, x\in[0,1],$$

whose absolute maximum follows the same probability law as

$$\begin{aligned}
&\mathcal{L}\left\{h^{-1}\int K\left(s-u/h\right)dW_n\left(u\right)/\left(h^{-1}\int_{-1}^{1}K^2\left(u\right)du\right)^{1/2}, s\in\left[0,h^{-1}\right]\right\}\\
&=\mathcal{L}\left\{\int K\left(s-r\right)dW_n\left(r\right)/\left(\int_{-1}^{1}K^2\left(t\right)dt\right)^{1/2}, s\in\left[0,h^{-1}\right]\right\}.
\end{aligned}$$

Let

$$\zeta(s) = \int K(s-r)\, dW_n(r) \Big/ \left(\int_{-1}^{1} K^2(t)\, dt \right)^{1/2}, s \in \left[0, h^{-1}\right].$$

According to Equation (2.5) in Bickel and Rosenblatt (1973), the following proposition holds:

Proposition 4. *Under Assumptions (M2) and (M4), as $n \to \infty$,*

$$P\left[a_h \left\{ \sup_{s \in [0,h^{-1}]} |\zeta(s)| - b_h \right\} < z \right] \to \exp\{-2\exp(-z)\}, z \in \mathbb{R},$$

in which a_h and b_h are given in Theorem 1.

3.2 Case of the variance function

To prove Theorem 2, the estimation error $\hat{\sigma}^2_{\mathrm{SK}}(x) - \tilde{\sigma}^2_{\mathrm{K}}(x)$ is broken into three simpler parts. We begin by describing the spline space $\mathcal{H}_N^{(p-2)}$ and the representation of the spline estimators $\hat{m}_p(x)$ in Equation (2.1).

The space $\mathcal{H}_N^{(p-2)}$ is spanned linearly by B-spline basis $\{b_{j,p}\}_{j=1-p}^{N}$ introduced in de Boor (2001). Denote by $\|\phi\|_2$ the theoretical L^2 norm of a function ϕ on $[0,1]$, $\|\phi\|_2^2 = \int_0^1 \phi^2(x)dx$ and the empirical L^2 norm as $\|\phi\|_{2,n}^2 = n^{-1}\sum_{i=1}^{n} \phi^2(i/n)$. The rescaled B-spline basis for $\mathcal{H}_N^{(p-2)}$ is $\{B_{j,p}\}_{j=1-p}^{N}$, where $B_{j,p}(x) = b_{j,p}(x)\, \|b_{j,p}\|_2^{-1}$ with theoretical norm equal to 1, $1-p \le j \le N$.

The estimator $\hat{m}_p(x)$ in Equation (2.3) can then be expressed as

$$\hat{m}_p(x) = \sum\nolimits_{j=1-p}^{N} \hat{\lambda}_{j,p} B_{j,p}(x),$$

where the vector $\left(\hat{\lambda}_{1-p,p}, \ldots, \hat{\lambda}_{N,p}\right)^T$ is the solution of the following least-squares problem

$$\left(\hat{\lambda}_{1-p,p}, \ldots, \hat{\lambda}_{N,p}\right)^T = \operatorname*{argmin}_{\mathbb{R}^{N+p}} \sum_{i=1}^{n} \left\{ Y_i - \sum\nolimits_{j=1-p}^{N} \lambda_{j,p} B_{j,p}(i/n) \right\}^2. \tag{3.6}$$

Furthermore, write $\mathbf{Y}$ as the sum of signal vector $\mathbf{m}$ and error term $\mathbf{E}$, i.e., $\mathbf{Y} = \mathbf{m} + \mathbf{E}$, where $\mathbf{Y} = \{Y_1, \ldots, Y_n\}^T$,$\mathbf{m} = \{m(1/n), \ldots, m(n/n)\}^T$, and $\mathbf{E} = \{\sigma(1/n)\varepsilon_1, \ldots, \sigma(n/n)\varepsilon_n\}^T$. Projecting this relationship into the space $\mathcal{H}_N^{(p-2)}$, one obtains that

$$\hat{m}_p(x) = \tilde{m}_p(x) + \tilde{\varepsilon}_p(x),$$

where

$$\tilde{m}_p(x) = \sum_{j=1-p}^{N} \tilde{\lambda}_{j,p} B_{j,p}(x),\ \tilde{\varepsilon}_p(x) = \sum_{j=1-p}^{N} \tilde{a}_{j,p} B_{j,p}(x), \tag{3.7}$$

and the vectors $\left\{\tilde{\lambda}_{1-p,p}, \ldots, \tilde{\lambda}_{N,p}\right\}^T$ and $\{\tilde{a}_{1-p,p}, \ldots, \tilde{a}_{N,p}\}^T$ in Equation (3.7) are solutions of Equation (3.6) with Y_i replaced by $m(i/n)$ and $\sigma(i/n)\varepsilon_i$ respectively. Applying elementary algebra, one obtains that

$$\tilde{m}_p(x) = \{B_{1-p,p}(x), \ldots, B_{N,p}(x)\} \left(\mathbf{B}^T \mathbf{B}\right)^{-1} \mathbf{B}^T \mathbf{m}, \tag{3.8}$$

$$\tilde{\varepsilon}_p(x) = \{B_{1-p,p}(x), \ldots, B_{N,p}(x)\} \left(\mathbf{B}^T \mathbf{B}\right)^{-1} \mathbf{B}^T \mathbf{E}, \tag{3.9}$$

where the design matrix $\mathbf{B}$ is

$$\mathbf{B} = \begin{pmatrix} B_{1-p,p}\left(\frac{1}{n}\right) & \cdots & B_{N,p}\left(\frac{1}{n}\right) \\ \vdots & \ddots & \vdots \\ B_{1-p,p}(1) & \cdots & B_{N,p}(1) \end{pmatrix}_{n\times(N+p)}.$$

It is obvious that $\hat{\sigma}^2_{\mathrm{SK}}(x) - \tilde{\sigma}^2_{\mathrm{K}}(x)$ can be decomposed as

$$\begin{aligned} \hat{\sigma}^2_{\mathrm{SK}}(x) - \tilde{\sigma}^2_{\mathrm{K}}(x) &= \frac{\sum_{i=1}^n \tilde{K}_{\tilde{h}}(i/n - x)(I_{i,p} + II_{i,p} + III_{i,p})}{\sum_{i=1}^n \tilde{K}_{\tilde{h}}(i/n - x)} \\ &= \{I_1 + I_2 + I_3\} / \left\{n^{-1} \sum_{i=1}^n \tilde{K}_{\tilde{h}}(i/n - x)\right\} \end{aligned}$$

in which

$$I_1 = I_1(x) = n^{-1} \sum_{i=1}^n \tilde{K}_{\tilde{h}}(i/n - x)\, I_{i,p}, \tag{3.10}$$

$$I_2 = I_2(x) = n^{-1} \sum_{i=1}^n \tilde{K}_{\tilde{h}}(i/n - x)\, II_{i,p}, \tag{3.11}$$

$$I_3 = I_3(x) = n^{-1} \sum_{i=1}^n \tilde{K}_{\tilde{h}}(i/n - x)\, III_{i,p}, \tag{3.12}$$

$$I_{i,p} = \{m(i/n) - \tilde{m}_p(i/n)\}^2 + \tilde{\varepsilon}_p^2(i/n) + 2\{\tilde{m}_p(i/n) - m(i/n)\}\ \tilde{\varepsilon}_p(i/n),$$

$$II_{i,p} = -2\sigma(i/n)\varepsilon_i\ \tilde{\varepsilon}_p(i/n),\ III_{i,p} = \{m(i/n) - \tilde{m}_p(i/n)\}\sigma(i/n)\varepsilon_i.$$

Theorem 2 follows from the next three propositions whose proofs are in the Appendix.

Proposition 5. *Under Assumptions (E1)–(E7), as $n \to \infty$,*

$$\sup_{x\in[0,1]} |I_1(x)| = \mathcal{O}_p\left(N^{-2p} + n^{-1}N\right).$$

Proposition 6. *Under Assumptions (E2)–(E7), as $n \to \infty$,*

$$\sup_{x\in[0,1]} |I_2(x)| = \mathcal{O}_p\left(n^{-1}\tilde{h}^{-1/2}N^{3/2}\log^{1/2} n + \tilde{h}^{-1}n^{\beta-3/2}N^{3/2} + n^{\beta-3/2}N^{5/2}\right).$$

Proposition 7. *Under Assumptions (E1)–(E7), as $n \to \infty$,*

$$\sup_{x\in[0,1]} |I_3(x)| = \mathcal{O}_p\left(n^{-1/2}\tilde{h}^{-1/2}N^{1-p}\log^{1/2} n + \tilde{h}^{-1}n^{\beta-1}N^{1-p} + n^{\beta-1}N^{2-p}\right).$$

4. Implementation

In this section we describe detailed procedures for implementing the SCBs in Theorems 1 and 3 based on a data set $\{(i/n, Y_i)\}_{i=1}^n$ that follows model (1.1). It will be used throughout Section 5 for both simulated and real data examples. The implementation codes are written in R 3.03 and posted in the website: https://github.com/CaiLi16/SCBs-for-mean-and-variance-functions.

4.1 Implementing mean function SCB

As the default, we set $p = 2, \theta = 1$ in Assumption (M1). When constructing the SCB for the mean function $m(x)$ in model (1.1) according to Theorem 1, one chooses kernel function K and bandwidth h for computing $\hat{m}(x)$ and estimate the variance function $\sigma^2(x)$, and then plug in these estimates, as in Eubank and Speckman (1993), Hall and Titterington (1988), Härdle (1989), Xia (1998) and Wang and Yang (2009).

We choose the quartic kernel $K(u) = 15\left(1-u^2\right)^2 I\{|u| \le 1\}/16$ to satisfy Assumption (M4), and the following under smoothing bandwidths $h = h_{\mathrm{rot}} \times \log^{-1/5-\delta_1} n$ $(\delta_1 > 0)$ to satisfy Assumption (M5), where the rule-of-thumb bandwidth h_{rot} is from Equation (4.3) of Fan and Gijbels (1996):

$$h_{\mathrm{rot}} = \left\{ \frac{35\sum_{i=1}^n \left(Y_i - \sum_{k=0}^4 \widehat{a}_k\,(i/n)^k\right)^2}{n\sum_{i=1}^n \left(2\widehat{a}_2 + 6\widehat{a}_3\,(i/n) + 12\widehat{a}_4\,(i/n)^2\right)^2} \right\}^{1/5}, \tag{4.1}$$

in which $(\widehat{a}_k)_{k=0}^4 = \mathrm{argmin}_{(a_k)_{k=0}^4 \in \mathbb{R}^5} \sum_{i=1}^n \left(Y_i - \sum_{k=0}^4 a_k (i/n)^k\right)^2$. Note that h_{rot} has order $n^{-1/5}$ and h order $n^{-1/5}\log^{-1/5-\delta_1} n$, satisfying Assumption (M5). We have found in extensive simulations that $h = h_{\mathrm{rot}} \log^{-1/2} n$ works quite well and that is what we recommend.

The two step spline-kernel estimator $\hat{\sigma}^2_{\mathrm{SK}}(x)$ in Equation (2.2) is used for the variance function $\sigma^2(x)$, with detailed procedures introduced in Section 4.2.

The asymptotic $100(1-\alpha)\%$ SCB for the mean function is:

$$\hat{m}(x) \pm \hat{v}(x)\left[a_h + a_h^{-1}\left\{q_\alpha + 1/2\log\left(C_K/4\pi^2\right)\right\}\right], x \in \mathcal{I}_n, \tag{4.2}$$

with $\hat{v}(x) = \left\{n^{-1}h^{-1}\hat{\sigma}^2_{\mathrm{SK}}(x)\int_{-1}^1 K^2(v)\,dv\right\}^{1/2}$.

4.2 Implementing variance function SCB

To construct the SCB for $\sigma^2(x)$, we set the default values $p = 4, p_0 = 2, \theta_0 = 1$ in Assumptions (E1) and (E2) and let kernel $\tilde{K}(u) = 15\left(1-u^2\right)^2 I\{|u| \leq 1\}/16$ satisfying Assumption (E4), with bandwidth $\tilde{h} = h_{\mathrm{rot},\sigma}\log^{-1/5-\delta_2} n$ satisfying Assumption (E5), in which $\delta_2 > 0, h_{\mathrm{rot},\sigma}$ is as in (4.1) but with Y_i replaced by $\hat{e}_i^2 = \{Y_i - \hat{m}_p(i/n)\}^2$. Extensive simulation experiments show that $\tilde{h} = h_{\mathrm{rot},\sigma}\log^{-1/2} n$ works quite well and it is what we recommend.

According to Theorem 1 of Xue and Yang (2006), for any $m(x) \in C^p[0,1]$, $p \geq 2$, the optimal order of knots number N for $m(x)$ is $n^{1/(2p+1)}$, which is $n^{1/9}$ with $p = 4$. Denote the 'optimal' N by $\hat{N}^{\mathrm{opt}}$ which is the minimizer of the AIC defined below over integers in $[0.5N_r, \min\{5N_r, Tb\}]$, where $N_r = n^{1/9}$ and $Tb = n/4 - 1$ to ensure that $\hat{N}^{\mathrm{opt}}$ is of order $n^{1/9}$ and the total parameters in the least square estimation is less than $n/4$. This particular $\hat{N}^{\mathrm{opt}}$ satisfies Assumption (E7), but is of course not the only one. Let $\hat{Y}_i = \hat{m}_p(i/n)$ be the predictor of the i-th response Y_i and $q_n = (4+N)$ represent the number of parameters in (3.6). The AIC value corresponding to N is

$$\mathrm{AIC}(N) = \log \mathrm{MSE} + 2q_n/n, \quad \mathrm{MSE} = n^{-1}\sum_{i=1}^n (Y_i - \hat{Y}_i)^2. \tag{4.3}$$

To estimate the variance function $\sigma_0^2(x)$ of $\left\{(i/n, e_i^2)\right\}_{i=1}^n$, one uses the spline-kernel method just as described above. Specifically, denote $\hat{\sigma}^2_{\mathrm{S}}(x)$ as the spline estimator based on data set $\left\{(i/n, \hat{e}_i^2)\right\}_{i=1}^n$:

$$\hat{\sigma}^2_{\mathrm{S}}(x) = \{B_{1-p,p}(x), \ldots, B_{N,p}(x)\}\left(\mathbf{B}^T\mathbf{B}\right)^{-1}\mathbf{B}^T\left\{\hat{e}_1^2, \ldots, \hat{e}_n^2\right\}^T.$$

Let $\nabla_i = \left\{\hat{e}_i^2 - \hat{\sigma}_{\mathrm{S}}^2\left(i/n\right)\right\}^2$. Then the following estimator $\hat{\sigma}_0^2(x)$ is obtained:

$$\hat{\sigma}_0^2(x) = \frac{n^{-1}\sum_{i=1}^n \mathcal{K}_{h_{\sigma_0}}\left(i/n - x\right)\nabla_i}{n^{-1}\sum_{i=1}^n \mathcal{K}_{h_{\sigma_0}}\left(i/n - x\right)},$$

where h_{σ_0} is a under smoothing bandwidth, $h_{\sigma_0} = h_{\mathrm{rot},\sigma_0}\log^{-1/2} n$, in which $h_{\mathrm{rot},\sigma_0}$ is the rule-of-thumb bandwidth as in Subsection 4.1 with Y_i replaced by ∇_i. According to Fan and Gijbels (1996), one has $\sup_{x\in[0,1]}\left|\hat{\sigma}_0^2\left(x\right) - \sigma_0^2(x)\right| = o_p\left(1\right)$. The asymptotic $100(1-\alpha)\%$ SCB for variance function is:

$$\hat{\sigma}_{\mathrm{SK}}^2(x) \pm \hat{v}_0(x)\left[a_{\tilde{h}} + a_{\tilde{h}}^{-1}\left\{q_\alpha + 1/2\log\left(C_{\tilde{K}}/\left(4\pi^2\right)\right)\right\}\right], x \in \tilde{\mathcal{I}}_n, \tag{4.4}$$

with $\hat{v}_0(x) = \left(n^{-1}h^{-1}\hat{\sigma}_0^2(x)\int_{-1}^1 \tilde{K}^2\left(v\right)dv\right)^{1/2}$.

5. Empirical Studies

5.1 Monte Carlo examples

To investigate the finite-sample behavior of the proposed SCBs in Section 2, four cases in Table 1 are examined where

$$m_1(x) = \cos\left(3\pi x\right), m_2(x) = \exp\left(-32x^2\right),$$

$$\sigma_1(x) = 0.1\sin\left(2\pi x\right) + 0.2, \sigma_2(x) = \frac{\exp(x) - 0.9}{\exp(x) + 0.9},$$

and ε is distributed either as $N\left(0,1\right)$ or standardized t-distribution with freedom 10, i.e., $\varepsilon \sim 0.8^{1/2} * t_{10}$. The mean functions $m_1(x), m_2(x)$ resemble those in Eubank and Speckman (1993), but without periodicity. The sample size $n = 300, 600, 900$ while for the SCBs, the confidence level $1-\alpha = 0.95, 0.99$.

Table 1: Four Cases of Study

Case 1	Case 2	Case 3	Case 4
$m_1(x), \sigma_1(x)$	$m_1(x), \sigma_2(x)$	$m_2(x), \sigma_1(x)$	$m_2(x), \sigma_2(x)$

The coverage frequencies by SCBs defined in (4.2) for $m(x)$ are reported in Table 2, which are relative frequencies out of 2000 replications of coverage of the true curve at equally spaced points $\{x_j, j = 1, 2, \ldots, 400\}$ on $\mathcal{I}_n$. For comparison, the coverage frequencies from Eubank and Speckman (1993) are also listed in Table 2 and denoted as SCB-ES for simplicity. In all cases with $\varepsilon \sim N(0,1)$ and $\varepsilon \sim 0.8^{1/2} * t_{10}$ the coverage frequencies improve and approach the nominal level as the sample size n increases, which confirms Theorem 1. It is also evident that the SCBs in (4.2) perform far better than those in Eubank and Speckman (1993).

The coverage frequencies at equally spaced points $\{x_j, j = 1, 2, \ldots, 400\}$ on $\tilde{\mathcal{I}}_n$ by the SCB in (4.4) and the 'infeasible' SCB in (2.8) for $\sigma^2(x)$ are shown in Table 3. The coverage frequencies improve and approach the nominal levels as the sample size n increases for all cases, which confirms Theorem 3. Meanwhile, the coverage frequencies by the SCB and the 'infeasible' SCB are very close, which confirms the oracle efficiency in Theorem 2.

Figures 1 depicts the boxplots of $\triangle_n = \sqrt{n}\max_j \left|\tilde{\sigma}^2_{\mathrm{K}}(x_j) - \hat{\sigma}^2_{\mathrm{SK}}(x_j)\right|$ based on $\varepsilon \sim N(0,1)$ over 2000 replications, where $\{x_j, j = 1, 2, \ldots, 400\}$ are equally-spaced points on $\tilde{\mathcal{I}}_n$. The boxplot of $\triangle_n$ becomes narrower as the sample size n increases. That is, the difference between $\hat{\sigma}^2_{\mathrm{SK}}(x)$ and $\tilde{\sigma}^2_{\mathrm{K}}(x)$ is asymptotically of an order smaller than $n^{-1/2}$, which confirms Theorem 2. The scenario with $\varepsilon \sim 0.8^{1/2} * t_{10}$ is shown in Figure S.1 in the supplement.

To visualize the SCBs for the mean and variance functions, Figures 2 and 3 are created based on two samples of size 300 and 600 in Case 1 with $\varepsilon \sim N(0,1)$. Each has the center solid line as the true curve, center dotted line the estimated curve and the upper and lower dashed lines the SCB. As expected, in all the figures the SCBs for $n = 600$ are thinner and fit better than those for $n = 300$. Figures S.2 and S.3 in the supplement show the SCBs for the mean and variance functions with $\varepsilon \sim 0.8^{1/2} * t_{10}$.

5.2 Real data examples

We have analyzed with our SCBs two data sets provided by Professor Jiang Yaodong's research group at China University of Mining and Technology, which are available from us upon request. The data are actual strata pressure records in May 2013, from the Bulianta Coal Mine located in Ordos City, Inner Mongolia, China. Information on strata pressure behavior, range and pressure periodicity in front of working face is important for the coal mine industry to improve underground mining safety and precision, by preparing the roof support design to prevent accidents caused by sudden increase of strata pressure, see related paper Ju and Xu (2013) and Qian et al. (2010).

Strata pressure is the vertical stress on the coal seam roof in front of the working face with unit $\mathrm{KN/m^2}$ (working face is the underground location where miners peel coal from the coal wall mechanically). The pressure sensors are placed at the top of hydraulic supports in front of working face, and collect data with a record interval of 0.80m. That is, during the mining process, once the hydraulic support has moved forward 0.80m, a pressure sensor records a mine pressure. The propulsion range of the hydraulic support is from 295.5m to 705.1m (i.e., the sample size n is 513). We have chosen from more than 20 pressure records two representative sets for analysis, named records 1 and 2 respectively.

A potential concern is whether the independence assumption on errors $\varepsilon_i, 1 \leq i \leq n$, would be satisfied in real data applications. Although it is impossible to "prove" such independence for any real data, coal mine experts generally believe that measurement errors in strata pressure are caused by random geological conditions and systematic errors of the sensors and therefore independent. In addition, we plotted for record 1 the autocorrelation function (acf) of residuals $\hat{\varepsilon}_i = \{y_i - \hat{m}(X_i)\}/\hat{\sigma}_{\mathrm{SK}}(X_i), 1 \leq i \leq n$, and of $\{|\hat{\varepsilon}_i|\}_{i=1}^n, \{\hat{\varepsilon}_i^2\}_{i=1}^n, \{\hat{\varepsilon}_i^4\}_{i=1}^n$ in Figure S.4 in the supplement. An inspection of these plots shows that the percentage of acfs exceeding the 95% confidence limits are either $1/40 = 0.025$ or $2/40 = 0.05$, hence the null hypothesis of zero autocorrelation is not rejected, for $\hat{\varepsilon}_i, \{|\hat{\varepsilon}_i|\}_{i=1}^n, \{\hat{\varepsilon}_i^2\}_{i=1}^n, \{\hat{\varepsilon}_i^4\}_{i=1}^n$, lending further support of no violation of the independence assumption. For the other records, the same acf pattern is observed.

Figure 4 shows the plots for record 1 the SCB (solid) computed according to (4.2) for the mean function $m(x)$, kernel estimate $\hat{m}(x)$ (dashed) with confidence level 95% and 15.3% respectively. According to the theory of strata pressure, pp. 65-73 of Qian et al. (2010), the pressure behavior is periodic. We have therefore proposed the null hypothesis curve $m(x) = a_0 + \sum_{k=1}^5 \{a_k \sin(k\omega x) + b_k \cos(k\omega x)\}$ to be tested by the SCB for the mean function $m(x)$. Since the lowest confidence level of SCB containing the null curve is 15.3%, one retains the null hypothesis with the p-value $= 0.847$. Likewise, Figure S.5 in the supplement plots for record

2 the SCB (solid) computed according to (4.2) for the mean function, the null hypothesis curve $m(x) = a_0 + \sum_{k=1}^{5}\{a_k \sin(k\omega x) + b_k \cos(k\omega x)\}$ (thick solid) and kernel estimate $\hat{m}(x)$ (dashed) with confidence level 95% and 50.8% respectively. As for record 1, one retains the null hypothesis with the p-value $= 0.492$. We note that the estimated periodicity for records 1 and 2 are $\omega = 0.0144$ and $\omega = 0.01566$ respectively, which are rather close. Further investigation may lead to conclusive evidence for a general periodicity model for the mean function $m(x)$ with a common ω. Lastly, one observes with 95% confidence that the strata pressure range is [233.5KN/m^2, 320.3KN/m^2] (the lowest and highest values of the 95% SCB curves) for record 1, [230.8KN/m^2, 370.1KN/m^2] for record 2, also quite close.

In Figure 5 the center dashed line is the spline kernel estimator $\hat{\sigma}^2_{\mathrm{SK}}(x)$ for $\sigma^2(x)$ and the upper/lower solid lines represent the SCB for $\sigma^2(x)$. The SCB is used to detect heteroscedasticity in the data with the null hypothesis H_0: $\sigma^2(x) \equiv \sigma^2$. Since the lowest confidence level of SCB containing the horizontal line $\hat{\sigma}^2 = 1/n\sum_{i=1}^{n}\hat{e}_i^2$ is 45.5%, where $\hat{\sigma}^2$ is a consistent estimate of σ^2 under H_0, one retains the null hypothesis of homoscedasticity with the p-value $= 0.545$. In contrast, Figure S.6 in the supplement shows that for record 2, the null hypothesis of homoscedasticity is rejected with the p-value $= 0.0024$.

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Online Supplementary Materials

The online supplement contains theoretical proofs for Lemmas A.2, A.6, A.7, A.8 and Proposition 1 and some figures for the simulated and real data examples.

Appendix

A.1. Preliminaries

In the following, we use c and C to denote any positive constants in the generic sense. We will need the following lemmas to prove our main results.

Lemma A.1. *(Komlós, Major, and Tusnády, 1976, Theorem 4) Suppose* $\varepsilon_i, 1 \leq i < \infty$, *are i.i.d. r.v.'s with* $\mathrm{E}\,\varepsilon_1 = 0$ *and* $\mathrm{E}\,\varepsilon_1^2 = 1$. *Let* $H(x) > 0, x > 0$ *be a monotone increasing and continuous function such that for constants* $\delta > 0, x_0 > 0$, $x^{-3-\delta}H(x)$ *is monotone increasing for* $x > x_0$, *and* $x^{-1}\log H(x)$ *is monotone decreasing for* $x > x_0$. *Define* K_n *by the equation* $H(K_n) = n$. *If* $\mathrm{E}\,H(|\xi_1|) < \infty$, *then there exist constants* $C_1, C_2, a > 0$ *depending only on the distribution of* ξ_1 *and a sequence* $\{Z_i\}_{i=1}^n$ *of i.i.d. r.v.'s with standard normal distribution such that for any* t, $t > K_n, t^2/\log H(t) < C_1 n$,

$$P\left\{\max_{1\leq l\leq n} |S_l - W_l| > t\right\} \leq C_2 n \left\{H(at)\right\}^{-1},$$

where $S_l = \sum_{i=1}^l \varepsilon_i$ *and* $W_l = \sum_{i=1}^l Z_i$.

Lemma A.2. *Assumption (M3) holds under Assumption (M3').*

Proof of Lemma A.2. See the online supplement.

In the next following lemmas, we use $\mathcal{O}(1)$ to mean 'bounded for any fixed $x \in [0,1]$', $\mathcal{O}_{a.s.}(1)$ $[o_{a.s.}(1)]$ to mean 'bounded [tends to 0] almost surely for any fixed $x \in [0,1]$' and $U_p(1)$ $[u_p(1)]$ to mean 'bounded [tends to 0] in probability uniformly for any $x \in [0,1]$'. Let S_l be as given in Lemma A.1 and $W_{l,n} = \sum_{i=1}^l Z_{in}$ for independent standard normal variables $\{Z_{in}\}_{i=1}^n$ as in Equation (S1.1) in the supplement. Then we have the following lemma.

Lemma A.3. *Under Assumption (M3), as* $n \to \infty$, S_n *and* $W_{n,n}$ *satisfy*

$$\left|n^{-1}S_n - n^{-1}W_{n,n}\right| = \mathcal{O}_{a.s.}\left(n^{\beta-1}\right).$$

Proof of Lemma A.3. See the proof of Lemma A.5 in Cao et al. (2012).

For any Lebesgue measurable function $\phi(x)$ on $[0,1]$, denote $\|\phi(x)\|_\infty = \sup_{x\in[0,1]} |\phi(x)|$ and denote a class of Lipschitz continuous functions by $\mathrm{Lip}([0,1], C) = \{\phi(x) \,|\, |\phi(x) - \phi(x')| \le C|x - x'|, \forall x, x' \in [0,1], C > 0\}$.

Lemma A.4. *Under Assumption (E1), there exists a constant $C_p > 0$ with $p > 1$, such that for any $m \in C^p[0,1]$, there exists a function $g \in \mathcal{H}_N^{(p-2)}$ for which $\|g - m\|_\infty \le C_p N^{-p}$ and $g - m \in \mathrm{Lip}\left([0,1], C_p N^{1-p}\right)$. Furthermore, the function $\tilde{m}_p(x)$ given in (3.8) satisfies:*

$$\|\tilde{m}_p(x) - m(x)\|_\infty \le C \inf_{g\in\mathcal{H}_N^{(p-2)}} \|g(x) - m(x)\|_\infty = \mathcal{O}\left(N^{-p}\right).$$

Proof of Lemma A.4. See de Boor (2001) Theorem 6 on p.149 and Theorem 26 on p.155 for the detailed proofs.

The following result is based on Lemma A.3 of Song and Yang (2009) and Lemma A.1 of Xue and Yang (2006) in which $\|\boldsymbol{\varsigma}\|$ represents the Euclidean norm for any vector $\boldsymbol{\varsigma}$.

Lemma A.5. *Under Assumption (E7), there exists constants c and C independent of n such that for any vector $\eta = \{\eta_{1-p}, \ldots, \eta_N\}^T \in \mathbb{R}^{N+p}$,*

$$c\sum_{j=1-p}^{N} \eta_j^2 \le \left\| \sum_{j=1-p}^{N} \eta_j B_{j.p}(x) \right\|_2^2 \le C \sum_{j=1-p}^{N} \eta_j^2, \tag{A.1}$$

and for large n,

$$c\|\eta\|^2 \le \eta^T \left(n^{-1}\mathbf{B}^T\mathbf{B}\right)^{-1} \eta \le C\|\eta\|^2. \tag{A.2}$$

Lemma A.6. *Under Assumptions (E2) and (E7), as $n \to \infty$, the function $\tilde{\varepsilon}_p(x)$ given in (3.7) satisfies $\|\tilde{\varepsilon}_p(x)\|_\infty = \mathcal{O}_p\left(n^{-1/2}N^{1/2}\right)$.*

Proof of Lemma A.6. See the online supplement.

Lemma A.7. *Under Assumptions (E2), (E4) and (E6), we denote $\|\varphi_n\|_\infty = \rho_n$ for any function φ_n with domain $[0,1]$. Then for any i.i.d. $N(0,1)$ variables $\{Z_{in}\}_{i=1}^n$ satisfying (E6), as $n \to \infty$,*

$$\sup_{x\in[0,1]} \left| n^{-1}\sum\nolimits_{i=1}^{n} \tilde{K}_{\tilde{h}}(i/n - x)\,\sigma(i/n)\,\varphi_n(i/n)\,Z_{in} \right| = \mathcal{O}_p\left(n^{-1/2}\tilde{h}^{-1/2}\rho_n \log^{1/2} n\right).$$

Proof of Lemma A.7**.** See the online supplement.

Lemma A.8**.** *Under Assumptions (E2), (E4) and (E6), for any function* $\varphi_n \in \mathrm{Lip}\left([0,1], C_{\varphi,n}\right)$ *and* $\|\varphi_n\|_\infty = \rho_n$*, as* $n \to \infty$,

$$\sup_{x\in[0,1]} \left| n^{-1} \sum\nolimits_{i=1}^{n} \tilde{K}_{\tilde{h}}\left(i/n - x\right) \sigma\left(i/n\right) \varphi_n\left(i/n\right) \varepsilon_i \right|$$
$$= \mathcal{O}_p\left(n^{-1/2} \tilde{h}^{-1/2} \rho_n \log^{1/2} n + n^{\beta-1} C_{\varphi,n} + n^{\beta-1} \tilde{h}^{-1} \rho_n \right).$$

Proof of Lemma A.8**.** See the online supplement.

A.2. Proofs of the Propositions and Theorems

Proof of Proposition 1. See the online supplement.

Proof of Proposition 2. For $x \in \mathcal{I}_n$, $A_n(x)$ in (3.1) can be written as

$$A_n(x) = \sum_{i=1}^{n} \int_{(i-1)/n}^{i/n} K_h\left(u - x\right) \left\{m\left(u\right) - m(x)\right\} du +$$
$$\sum_{i=1}^{n} \int_{(i-1)/n}^{i/n} \left[K_h\left(i/n - x\right) \left\{m\left(i/n\right) - m(x)\right\} - K_h\left(u - x\right) \left\{m\left(u\right) - m(x)\right\}\right] du.$$

Notice that

$$\begin{aligned} \left| \sum_{i=1}^{n} \int_{(i-1)/n}^{i/n} K_h\left(u - x\right) \left\{m\left(u\right) - m(x)\right\} du \right| &= \left| \int_0^1 K_h\left(u - x\right) \left\{m\left(u\right) - m(x)\right\} du \right| \\ &= \left| \int_{-1}^{1} K\left(v\right) \left\{m\left(x + hv\right) - m(x)\right\} dv \right| \\ \leq \left| \int_{-1}^{1} K\left(v\right) \left\{ m^{(1)}(x) hv + \frac{1}{2} m^{(2)}(x) \left(hv\right)^2 + \cdots + \frac{1}{(p-1)!} m^{(p-1)}\left(x\right) \left(hv\right)^{p-1} \right\} dv \right| & \\ + \int_{-1}^{1} \frac{c}{(p-1)!} \left|K\left(v\right)\right| \left(hv\right)^{\theta+p-1} dv = \mathcal{O}\left(h^{\theta+p-1}\right). & \qquad \text{(A.3)} \end{aligned}$$

Meanwhile,

$$\sum_{i=1}^{n}\int_{(i-1)/n}^{i/n}\left[K_h\left(i/n-x\right)\left\{m\left(i/n\right)-m(x)\right\}-K_h\left(u-x\right)\left\{m\left(u\right)-m(x)\right\}\right]du$$
$$=\sum_{i=1}^{n}\int_{(i-1)/n}^{i/n}\left[K_h\left(i/n-x\right)\left\{m\left(i/n\right)-m(x)\right\}-K_h\left(i/n-x\right)\left\{m\left(u\right)-m(x)\right\}\right]du$$
$$+\sum_{i=1}^{n}\int_{(i-1)/n}^{i/n}\left[K_h\left(i/n-x\right)\left\{m\left(u\right)-m(x)\right\}-K_h\left(u-x\right)\left\{m\left(u\right)-m(x)\right\}\right]du. \tag{A.4}$$

Thus, the first term in Equation (A.4) equals

$$\left|\sum_{i=1}^{n}\int_{(i-1)/n}^{i/n}K_h\left(i/n-x\right)\left\{m\left(i/n\right)-m(u)\right\}du\right|$$
$$\leq h^{-1}\left\|K\right\|_\infty\left\|m^{(1)}(x)\right\|_\infty\sum_{i=1}^{n}\int_{(i-1)/n}^{i/n}\left|i/n-u\right|du=\mathcal{O}\left(n^{-1}h^{-1}\right) \tag{A.5}$$

and the second term in Equation (A.4) equals

$$\left|\sum_{i=1}^{n}\int_{(i-1)/n}^{i/n}\left\{K_h\left(i/n-x\right)-K_h\left(u-x\right)\right\}\left\{m\left(u\right)-m(x)\right\}du\right|$$
$$\leq\sum_{i=1}^{n}\int_{(i-1)/n}^{i/n}\left|\left\{K_h\left(i/n-x\right)-K_h\left(u-x\right)\right\}\left\{m\left(u\right)-m(x)\right\}\right|du$$
$$\leq C\left\|m^{(1)}\right\|_\infty\left\|K^{(1)}\right\|_\infty h\sum_{i=1}^{n}\int_{(i-1)/n}^{i/n}h^{-2}\left|i/n-u\right|du=\mathcal{O}\left(n^{-1}h^{-1}\right). \tag{A.6}$$

Collecting (A.3), (A.4), (A.5) and (A.6) establishes Proposition 2.

Proof of Proposition 3(a). Equations (3.2) and (3.3) imply that $\sup_{x\in[0,1]}|B_n(x)$ $-B_{n1}(x)|$ can be written as

$$\sup_{x\in[0,1]}\left|n^{-1}\sum\nolimits_{i=1}^{n}K_h\left(x-i/n\right)\sigma\left(i/n\right)\left\{S_i-W_{i,n}+W_{i-1,n}-S_{i-1}\right\}\right|$$
$$\leq\sup_{x\in[0,1]}\left|\Delta_n(x)\right|+\sup_{x\in[0,1]}\left|n^{-1}K_h\left(x-1\right)\sigma\left(n/n\right)\left\{S_n-W_{n,n}\right\}\right|, \tag{A.7}$$

in which $\Delta_n(x)$ equals

$$n^{-1}\sum\nolimits_{i=1}^{n-1}\left\{K_h\left(x-i/n\right)\sigma\left(i/n\right)-K_h\left(x-\left(i+1\right)/n\right)\sigma\left(\left(i+1\right)/n\right)\right\}\left(S_i-W_{i,n}\right),$$

and $\{S_i\}_{i=1}^n$ and $\{W_{i,n}\}_{i=1}^n$ are in Equation (S1.1) in the supplement with $S_0 = 0, W_{0,n} = 0$. Under Assumption (M2), one obtains that $\max_{1\leq i\leq n-1} |\sigma((i+1)/n) -\sigma(i/n)| = \mathcal{O}\left(n^{-1}\right)$. Thus, for n large enough,

$$\begin{aligned}
&\max_{1\leq i\leq n-1} \sup_{x\in[0,1]} |\{K_h(x-i/n)\sigma(i/n) - K_h(x-(i+1)/n)\sigma((i+1)/n)\}| \\
&= \max_{1\leq i\leq n-1} \sup_{x\in[0,1]} \left|\sigma(i/n)\left\{K_h(x-i/n) - K_h(x-(i+1)/n) + \mathcal{O}\left(n^{-1}h^{-1}\right)\right\}\right| \\
&\leq C_\sigma \left\|K^{(1)}\right\|_\infty h^{-2}n^{-1} = \mathcal{O}\left(n^{-1}h^{-2}\right),
\end{aligned}$$

which along with Assumption (M3) implies that

$$\begin{aligned}
\sup_{x\in[0,1]} |\Delta_n(x)| &\leq n^{-1}([2nh]+1) \max_{1\leq i\leq n-1} |S_i - W_{i,n}| \sup_{x\in[0,1]} \max_{1\leq i\leq n-1} \\
&\quad |K_h(x-i/n)\sigma(i/n) - K_h(x-(i+1)/n)\sigma((i+1)/n)| \\
&= U_p\left(n^{\beta-1}h^{-1}\right). \qquad (A.8)
\end{aligned}$$

Meanwhile, according to Lemma A.3,

$$\begin{aligned}
\left|n^{-1}K_h(x-1)\sigma(n/n)\{S_n - W_{n,n}\}\right| &\leq h^{-1}C_\sigma \|K\|_\infty n^{-1}|S_n - W_{n,n}| \\
&= \mathcal{O}_{a.s.}\left(n^{\beta-1}h^{-1}\right). \qquad (A.9)
\end{aligned}$$

Then Equations (A.7), (A.8) and (A.9) imply Proposition 3(a).

Proof of Proposition 3(b). According to Equations (3.3) and (3.4), $B_{n1}(x) - B_{n2}(x)$ is the Gaussian process with mean 0 and variance

$$\begin{aligned}
&\mathrm{E}\left\{n^{-1}\sum\nolimits_{i=1}^n K_h(i/n-x)\{\sigma(i/n)-\sigma(x)\}Z_{in}\right\}^2 \\
&= n^{-2}\sum\nolimits_{i=1}^n K_h^2(i/n-x)\{\sigma(i/n)-\sigma(x)\}^2 \\
&\leq n^{-2}h^{-2}\|K\|_\infty^2 \left\|\sigma^{(1)}(x)\right\|_\infty^2 h^2([2nh]+1) \leq Cn^{-1}h.
\end{aligned}$$

Using the tail property of the normal distribution and applying the discretization method as in the proof of Lemma A.7 in the supplement, one immediately obtains that

$$\sup_{x\in[0,1]} |B_{n1}(x) - B_{n2}(x)| = \mathcal{O}_p\left(n^{-1/2}h^{1/2}\log^{1/2} n\right),$$

which completes the proof of Proposition 3(b).

Proof of Proposition 3(c). The $B_{n2}(x)$ in (3.4) can be written as the Gaussian process $n^{-1/2}\sum_{i=1}^{n} K_h\left(x-i/n\right)\sigma\left(x\right)\left\{W_{i,n}-W_{i-1,n}\right\}$. Meanwhile note that, under Assumption (M4), $K_h\left(x-u\right)=0$ for $x\in\mathcal{I}_n=[h,1-h]$ and $u\in(-\infty,0]\cup[1,+\infty)$. Thus, for $x\in\mathcal{I}_n$,

$$\begin{aligned}
B_{n2}(x) &= n^{-1/2}\sum\nolimits_{i=1}^{n} K_h\left(x-i/n\right)\sigma(x)\left\{W_{i,n}-W_{i-1,n}\right\} \\
&= n^{-1/2}\sigma(x)\sum\nolimits_{i=1}^{n}\int_{(i-1)/n}^{i/n} K_h\left(x-u\right)dW_n\left(u\right) \\
&+ n^{-1/2}\sigma(x)\sum\nolimits_{i=1}^{n}\int_{(i-1)/n}^{i/n}\left[K_h\left(x-i/n\right)-K_h\left(x-u\right)\right]dW_u\left(u\right) \\
&= \int K_h\left(x-u\right)dW_n\left(u\right)+n^{-1/2}\sigma\left(x\right)\sum\nolimits_{i=1}^{n}\int_{(i-1)/n}^{i/n}\left\{K_h\left(x-i/n\right)\right. \\
&\left.-K_h\left(x-u\right)\right\}dW_u\left(u\right)=B_{n3}(x)+\kappa_n(x), \qquad \text{(A.10)}
\end{aligned}$$

where $\kappa_n(x)=n^{-1/2}\sigma(x)\sum_{i=1}^{n}\int_{(i-1)/n}^{i/n}\left\{K_h\left(x-i/n\right)-K_h\left(x-u\right)\right\}dW_u\left(u\right)$ is a Gaussian process with mean 0 and variance

$$\begin{aligned}
\mathrm{E}\,\kappa_n^2(x) &= \mathrm{E}\left[n^{-1/2}\sigma(x)\sum_{i=1}^{n}\int_{(i-1)/n}^{i/n}\left\{K_h\left(x-i/n\right)-K_h\left(x-u\right)\right\}dW_n\left(u\right)\right]^2 \\
&= n^{-1}\sigma^2(x)\sum_{i=1}^{n}\mathrm{E}\left[\int_{(i-1)/n}^{i/n}\left\{K_h\left(x-i/n\right)-K_h\left(x-u\right)\right\}dW_n\left(u\right)\right]^2 \\
&= n^{-1}\sigma^2(x)\sum_{i=1}^{n}\int_{(i-1)/n}^{i/n}\left\{K_h\left(x-i/n\right)-K_h\left(x-u\right)\right\}^2du.
\end{aligned}$$

Thus the variance of $\kappa_n(x)$ is bounded by

$$C_\sigma^2 n^{-1}h^{-4}\sum_{i=1}^{n}\int_{(i-1)/n}^{i/n}\left\|K^{(1)}\right\|_\infty^2\left|u-i/n\right|^2du\le C_\sigma^2\left\|K^{(1)}\right\|_\infty^2 n^{-3}h^{-4}\le Cn^{-3}h^{-4}.$$

Similarly, using the tail property of the normal distribution and applying the discretization method, one obtains that

$$\sup_{x\in[0,1]}\left|\kappa_n(x)\right|=\mathcal{O}_{a.s}\left(n^{-3/2}h^{-2}\log^{1/2}n\right),$$

which together with (A.10) implies Proposition 3(c).

Proof of Proposition 3(d). The $B_{n3}(x)$ in (3.5) is the Gaussian process with mean 0 and variance

$$\begin{aligned}\mathrm{E}\,B_{n3}^2(x) &= n^{-1}\sigma^2(x)\,\mathrm{E}\left\{\int K_h\left(x-u\right)dW_n\left(u\right)\right\}^2 \\ &= n^{-1}\sigma^2(x)\int K_h^2\left(x-u\right)du = n^{-1}h^{-1}\sigma^2(x)\int_{-1}^{1}K^2\left(u\right)du.\end{aligned}$$

Once again using the tail property of the normal distribution and applying the discretization method, one obtains Proposition 3(d).

Proof of Proposition 5. Obviously, $I_1(x)$ in (3.10) is bounded by

$$\begin{aligned}&n^{-1}\sum\nolimits_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n-x\right)\left[2\left\{m\left(i/n\right)-\tilde{m}_p\left(i/n\right)\right\}^2+2\,\tilde{\varepsilon}_p^2\left(i/n\right)\right] \\ &\le 2n^{-1}\sum\nolimits_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n-x\right)\left\{\left\|m(x)-\tilde{m}_p\left(x\right)\right\|_\infty^2+\ \left\|\tilde{\varepsilon}_p(x)\right\|_\infty^2\right\} \\ &\le U_p\left(N^{-2p}+n^{-1}N\right).\end{aligned}$$

Thus Proposition 5 is proved.

Proof of Proposition 6. From Equation (3.7), one knows that $I_2(x)$ in (3.11) equals $2n^{-1}\sum_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n-x\right)\sigma\left(i/n\right)\varepsilon_i\sum_{j=1-p}^{N}\tilde{a}_{j,p}B_{j,p}(i/n)$. Hence the Cauchy- Schwartz inequality implies that

$$|I_2(x)| \le 2\left[\sum\nolimits_{j=1-p}^{N}\tilde{a}_{j,p}^2\sum\nolimits_{j=1-p}^{N}\left\{n^{-1}\sum\nolimits_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n-x\right)\sigma\left(i/n\right)\varepsilon_iB_{j,p}\left(i/n\right)\right\}^2\right]^{1/2}. \tag{A.11}$$

According to Equation (A.1) in Lemma A.5

$$c\left\|\sum\nolimits_{j=1-p}^{N}\tilde{a}_{j,p}B_{j,p}(x)\right\|_2^2 \le \sum\nolimits_{j=1-p}^{N}\tilde{a}_{j,p}^2 \le C\left\|\sum\nolimits_{j=1-p}^{N}\tilde{a}_{j,p}B_{j,p}(x)\right\|_2^2.$$

Hence applying Equation (3.7) and Lemma A.6, one obtains that

$$\sum\nolimits_{j=1-p}^{N}\tilde{a}_{j,p}^2 = \mathcal{O}_p\left(n^{-1}N\right). \tag{A.12}$$

Applying Lemma A.8 with $\varphi_n(x) = B_{j,p}(x)$, $C_{\varphi,n} = \mathcal{O}\left(N^{3/2}\right)$ and $\rho_n = \mathcal{O}\left(N^{1/2}\right)$, one has that

$$\begin{aligned}&\sup_{x\in[0,1]}\left|n^{-1}\sum\nolimits_{j=1-p}^{N}\tilde{K}_{\tilde{h}}\left(i/n-x\right)\sigma\left(i/n\right)\varepsilon_iB_{j,p}\left(i/n\right)\right| \\ &= \mathcal{O}_p\left(n^{-1/2}\tilde{h}^{-1/2}N^{1/2}\log^{1/2}n+\tilde{h}^{-1}n^{\beta-1}N^{1/2}+n^{\beta-1}N^{3/2}\right).\end{aligned} \tag{A.13}$$

Putting (A.12) and (A.13) together one obtains that

$$\sup_{x\in[0,1]}\left[\sum\nolimits_{j=1-p}^{N}\tilde{a}_{j,p}^{2}\sum\nolimits_{j=1-p}^{N}\left\{n^{-1}\sum\nolimits_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n-x\right)\sigma\left(i/n\right)\varepsilon_{i}B_{j,p}\left(i/n\right)\right\}^{2}\right]^{1/2}$$
$$=\mathcal{O}_{p}\left(n^{-1}\tilde{h}^{-1/2}N^{3/2}\log^{1/2}n+\tilde{h}^{-1}n^{\beta-3/2}N^{3/2}+n^{\beta-3/2}N^{5/2}\right),$$

which together with (A.11) implies Proposition 6.

Proof of Proposition 7. For I_3 in (3.12), one obtains that

$$\begin{aligned}|I_{3}(x)| &= \left|n^{-1}\sum\nolimits_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n-x\right)\left\{m\left(i/n\right)-\tilde{m}_{p}\left(i/n\right)\right\}\sigma\left(i/n\right)\varepsilon_{i}\right| \\ &\leq \left|n^{-1}\sum\nolimits_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n-x\right)\left\{m\left(i/n\right)-g\left(i/n\right)\right\}\sigma\left(i/n\right)\varepsilon_{i}\right|+ \\ &\quad \left|n^{-1}\sum\nolimits_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n-x\right)\left\{g\left(i/n\right)-\tilde{m}_{p}\left(i/n\right)\right\}\sigma\left(i/n\right)\varepsilon_{i}\right| \quad \text{(A.14)}\end{aligned}$$

in which $g\in\mathcal{H}_{N}^{(p-2)}$ satisfies $\|g-m\|_{\infty}\leq C_{p}N^{-p}, g-m\in\mathrm{Lip}\left([0,1],CN^{1-p}\right)$. Applying Lemma A.8 with $\varphi_{n}(x)=m(x)-g(x)$, one obtains that

$$\sup_{x\in[0,1]}\left|n^{-1}\sum\nolimits_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n-x\right)\left\{m\left(i/n\right)-g\left(i/n\right)\right\}\sigma\left(i/n\right)\varepsilon_{i}\right|$$
$$=U_{p}\left(n^{-1/2}\tilde{h}^{-1/2}N^{1-p}\log^{1/2}n+n^{\beta-1}N^{1-p}+n^{\beta-1}\tilde{h}^{-1}N^{-p}\right). \quad \text{(A.15)}$$

Since $g,\tilde{m}_{p}\in\mathcal{H}_{N}^{(p-2)}$, one can write $g(x)-\tilde{m}_{p}(x)=\sum_{J=1-p}^{N}\pi_{J,p}B_{J,p}(x)$. According to Lemmas A.4 and A.5, we have

$$\sum\nolimits_{J=1-p}^{N}\pi_{J,p}^{2}\leq C\left\|\sum\nolimits_{J=1-p}^{N}\pi_{J,p}B_{J,p}(x)\right\|_{2}^{2}\leq C\left\|g(x)-\tilde{m}_{p}(x)\right\|_{\infty}=U_{p}\left(N^{-2p}\right). \quad \text{(A.16)}$$

Equations (A.13) and (A.16) imply that

$$\sup_{x\in[0,1]}\left|n^{-1}\sum\nolimits_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n-x\right)\left\{g\left(i/n\right)-\tilde{m}_{p}\left(i/n\right)\right\}\sigma\left(i/n\right)\varepsilon_{i}\right|$$
$$\leq\sup_{x\in[0,1]}\left[\sum\nolimits_{J=1-p}^{N}\pi_{J,p}^{2}\sum\nolimits_{J=1-p}^{N}\left\{n^{-1}\sum\nolimits_{i=1}^{n}\tilde{K}_{\tilde{h}}\left(i/n-x\right)B_{J,p}\left(i/n\right)\sigma\left(i/n\right)\varepsilon_{i}\right\}^{2}\right]^{1/2}$$
$$=U_{p}\left(n^{-1/2}\tilde{h}^{-1/2}N^{1-p}\log^{1/2}n+\tilde{h}^{-1}n^{\beta-1}N^{1-p}+n^{\beta-1}N^{2-p}\right),$$

which together with (A.14) and (A.15) implies Proposition 7.

References

Angelini, C., De Canditiis, D. and Frédérique L. (2003). Wavelet regression estimation in nonparametric mixed effect models. *J. Multivar. Anal.* **85**, 267-291.

Bickel, P. and Rosenblatt, M. (1973). On some global measures of deviations of density function estimates. *Ann. Statist.* **31**, 1852-1884.

Brown, L. and Levine M. (2007). Variance estimation in nonparametric regression via the difference sequence method. *Ann. Statist.* **35**, 2219-2232.

Cai, T., Levine, M. and Wang, L. (2009). Variance function estimation in multivariate nonparametric regression with fixed design. *J. Multivar. Anal.* **1**, 126-136.

Cai, T., Low, M. and Ma, Z. (2014). Adaptive confidence bands for nonparametric regression functions. *J. Amer. Statist. Assoc.* **109**, 1054-1070.

Cai, L. and Yang, L. (2015). A smooth simultaneous confidence band for conditional variance function. *TEST* **24**, 632-655.

Cao, G., Wang, L., Li, Y. and Yang, L. (2016). Oracle-efficient confidence envelopes for covariance functions in dense functional data. *Statist. Sinica* **26**, 359-383.

Cao, G., Yang, L. and Todem, D. (2012). Simultaneous inference for the mean function based on dense functional data. *J. Nonparametr. Statist.* **24**, 359-377.

Degras, D. (2011). Simultaneous confidence bands for nonparametric regression with functional data. *Statist. Sinica* **21**, 1735-1765.

de Boor, C. (2001). *A Practical Guide to Splines*. Springer-Verlag, New York.

Donoho, D. and Johnstone, I. (1996). Neo-classical minimax problems, thresholding and adaptive function estimation. *Bernoulli* **13**, 7-9.

Eubank, R. and Speckman, P. (1993). Confidence bands in nonparametric regression. *J. Amer. Statist. Assoc.* **88**, 1287-1301.

Fan, J. and Gijbels, I. (1996). *Local Polynomial Modelling and Its Applications*. Chapman and Hall, London.

Gu, L., Wang, L., Härdle, W. and Yang, L. (2014). A simultaneous confidence corridor for varying coefficient regression with sparse functional data. *TEST* **23**, 806-843.

Gu, L. and Yang, L. (2015). Oracally efficient estimation for single-index link function with simultaneous confidence band. *Electronic J. Statist.* **9**, 1540-1561.

Härdle, W. (1989). Asmptotic maximal deviation of M-smoothers. *J. Multivar. Anal.* **29**, 163-

179.

Härdle, W. and Bowman A. (1988). Bootstrapping in nonparametric regression: Local adaptive smoothing and confidence bands. *J. Amer. Statist. Assoc.* **83**, 102-110.

Härdle, W. and Marron, J. (1991). Bootstrap simultaneous error bars for nonparametric regression. *Ann. Statist.* **19**, 778-796.

Hall, P. and Titterington, D. (1988). On confidence bands in nonparametric density estimation and regression. *J. Multivar. Anal.* **27**, 228-254.

Johnston, G. (1982). Probabilities of maximal deviations for nonparametric regression function estimates. *J. Multivar. Anal.* **12**, 402-414.

Ju, J. and Xu, J. (2013). Structural characteristics of key strata and strata behaviour of a fully mechanized longwall face with 7.0m height chocks. *Int. J. of Rock Mech. Min. Sci.* **58**, 46-54.

Komlós, J., Major, P. and Tusnády, G. (1976). An approximation of partial sums of independent RV's, and the sample DF. II. *Z. Wahrscheinlichkeitstheorie Verw. Gebiete* **34**, 33-58.

Keilegom, I. and Claeskens, G. (2003). Bootstrap confidence bands for regression curves and their derivatives. *Ann. Statist.* **31**, 1852-1884.

Levine M. (2006). Bandwidth selection for a class of difference-based variance estimators in the nonparametric regression: A possible approach. *Comput. Statist. Data Anal.* **50**, 3405-3431.

Ma, S., Yang, L. and Carroll, R. (2012). A simultaneous confidence band for sparse longitudinal regression. *Statist. Sinica* **22**, 95-122.

Nadaraya, E. (1964). On estimating regression. *Theory Probab. Appl.* **9**, 141-142.

Neumann, M. and Polzehl, J. (1998). Simultaneous bootstrap confidence bands in nonparametric regression. *J. Nonparametr. Statist.* **9**, 307-333.

Qian, M., Shi, P. and Xu, J. (2010). *Mining Pressure and Strata Control.* China University of Mining and Technology Press.

Song, Q. and Yang, L. (2009). Spline confidence bands for variance functions. *J. Nonparametr. Statist.* **5**, 589-609.

Stapleton, J. (2009). *Linear Statistical Models, Second Edition.* John Wiley & Sons, Hoboken, NJ.

Wang, J. (2012). Modelling time trend via spline confidence band. *Ann. Inst. Stat. Math.* **64**, 275-301.

Wang, L., Brown, L., Cai, T. and Levine, M. (2008). Effect of mean on variance function estimation in nonparametric regression. *Ann. Statist.* **36**, 646-664.

Wang, J., Cheng, F. and Yang, L. (2013). Smooth simultaneous confidence bands for cumulative distribution functions. *J. Nonparametr. Statist.* **25**, 395-407.

Wang, J., Liu, R., Cheng, F. and Yang, L. (2014). Oracally efficient estimation of autoregressive error distribution with simultaneous confidence band. *Ann. Statist.* **42**, 654-668.

Wang, J. and Yang, L. (2009). Polynomial spline confidence bands for regression curves. *Statist. Sinica* **19**, 325-342.

Wang W., Cheng, Y., Wang, H., Li, W. and Wang, L. (2015). Coupled disaster-causing mechanisms of strata pressure behavior and abnormal gas emissions in underground coal extraction. *Environ. Earth Sci.* **74**, 1-19.

Watson, G. (1964). Smoothing regression analysis. *Sankhya* **26**, 359-372.

Xia, Y. (1998). Bias-corrected confidence bands in nonparametric regression. *J. Roy. Statist. Soc. Ser. B* **60**, 797-811.

Xue, L and Yang, L. (2006). Additive coefficient modeling via polynomial spline. *Statist. Sinica* **16**, 1423-1446.

Zheng, S., Yang, L. and Härdle, W. (2014). A smooth simultaneous confidence corridor for the mean of sparse functional data. *J. Amer. Statist. Assoc.* **109**, 661-673.

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Table 2: Empirical coverage frequencies of the SCB in (4.2) and the SCB in Eubank and Speckman (1993) for $m(x)$ using 2000 replications with $\varepsilon \sim N(0,1)$ and $\varepsilon \sim 0.8^{1/2} * t_{10}$ respectively.

			$\varepsilon \sim N(0,1)$		$\varepsilon \sim 0.8^{1/2} * t_{10}$	
	n	$1-\alpha$	SCB	SCB-ES	SCB	SCB-ES
Case 1	300	0.950	0.940	0.832	0.945	0.828
		0.990	0.995	0.950	0.995	0.953
	600	0.950	0.962	0.810	0.947	0.828
		0.990	0.995	0.941	0.993	0.955
	900	0.950	0.952	0.818	0.959	0.815
		0.990	0.996	0.952	0.996	0.953
Case 2	300	0.950	0.951	0.863	0.959	0.852
		0.990	0.995	0.961	0.997	0.954
	600	0.950	0.961	0.840	0.960	0.836
		0.990	0.997	0.953	0.995	0.951
	900	0.950	0.962	0.817	0.959	0.828
		0.990	0.998	0.949	0.997	0.952
Case 3	300	0.950	0.964	0.858	0.966	0.866
		0.990	0.996	0.963	0.998	0.968
	600	0.950	0.966	0.846	0.968	0.861
		0.990	0.997	0.962	0.997	0.965
	900	0.950	0.970	0.850	0.966	0.851
		0.990	0.997	0.963	0.999	0.963
Case 4	300	0.950	0.960	0.916	0.960	0.914
		0.990	0.995	0.983	0.996	0.983
	600	0.950	0.956	0.905	0.950	0.903
		0.990	0.995	0.981	0.997	0.979
	900	0.950	0.957	0.891	0.956	0.900
		0.990	0.995	0.970	0.995	0.981

Table 3: Empirical coverage frequencies of the oracle SCB in (4.4) and the 'infeasible' SCB in (2.4) for $\sigma^2(x)$ using 2000 replications with $\varepsilon \sim N(0,1)$ and $\varepsilon \sim 0.8^{1/2} * t_{10}$ respectively.

			$\varepsilon \sim N(0,1)$		$\varepsilon \sim 0.8^{1/2} * t_{10}$	
	n	$1-\alpha$	oracle	infeasible	oracle	infeasible
Case 1	300	0.950	0.889	0.907	0.846	0.862
		0.990	0.959	0.969	0.939	0.953
	600	0.950	0.929	0.942	0.906	0.904
		0.990	0.982	0.986	0.973	0.972
	900	0.950	0.939	0.945	0.928	0.931
		0.990	0.990	0.990	0.983	0.985
Case 2	300	0.950	0.889	0.905	0.852	0.864
		0.990	0.952	0.964	0.945	0.947
	600	0.950	0.943	0.955	0.907	0.906
		0.990	0.984	0.988	0.970	0.966
	900	0.950	0.958	0.955	0.925	0.933
		0.990	0.991	0.992	0.981	0.983
Case 3	300	0.950	0.893	0.907	0.855	0.862
		0.990	0.960	0.969	0.945	0.953
	600	0.950	0.928	0.942	0.906	0.904
		0.990	0.983	0.986	0.975	0.972
	900	0.950	0.941	0.945	0.926	0.931
		0.990	0.989	0.990	0.981	0.985
Case 4	300	0.950	0.895	0.905	0.869	0.864
		0.990	0.958	0.964	0.953	0.947
	600	0.950	0.943	0.955	0.914	0.906
		0.990	0.986	0.988	0.973	0.966
	900	0.950	0.933	0.955	0.917	0.933
		0.990	0.989	0.992	0.983	0.983

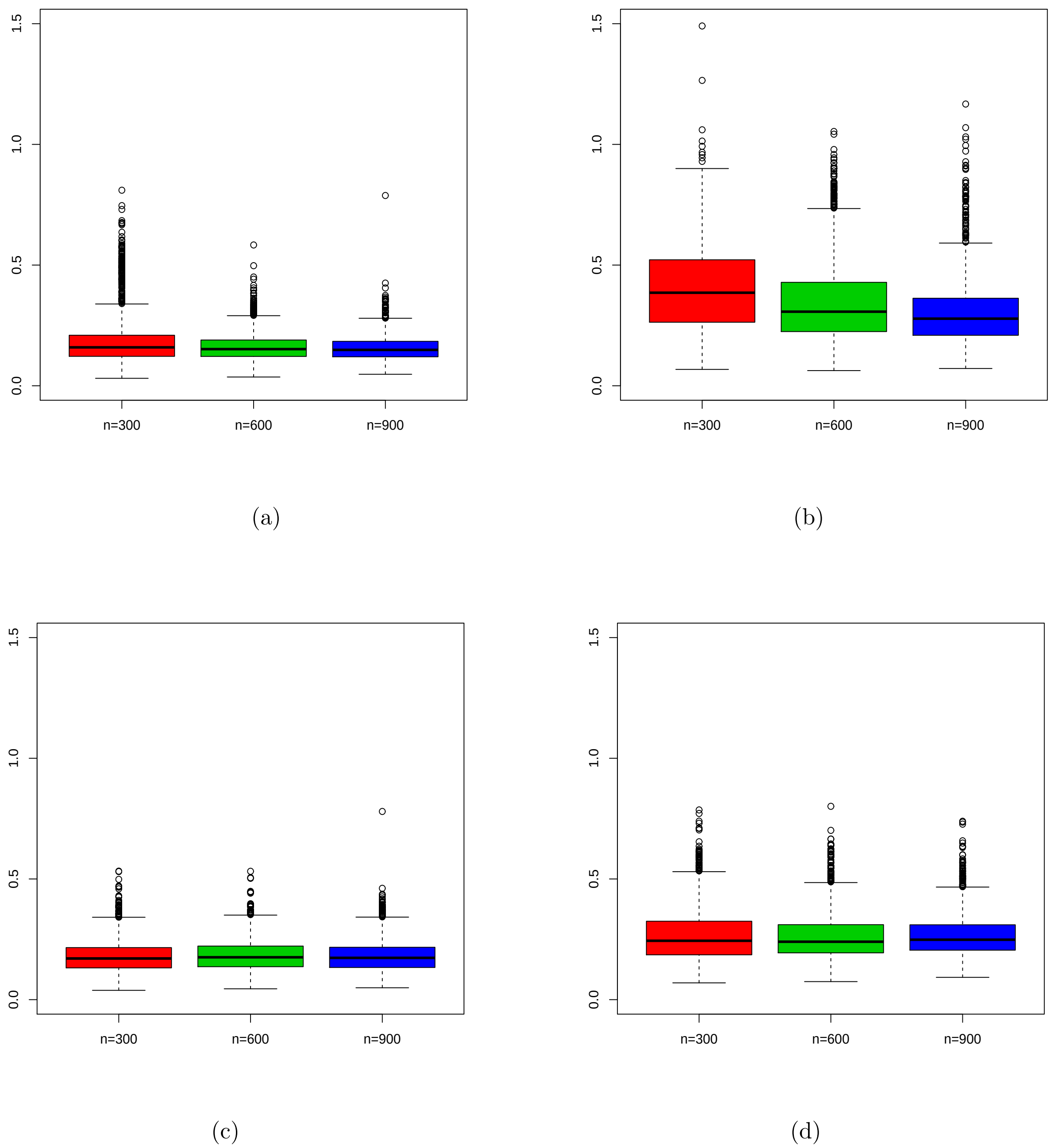
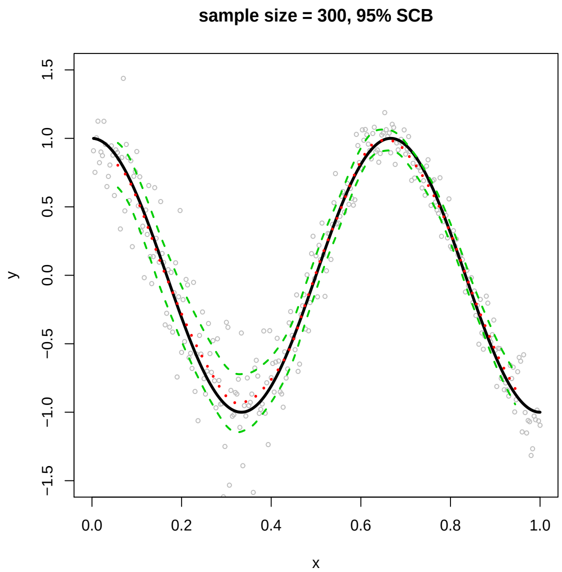
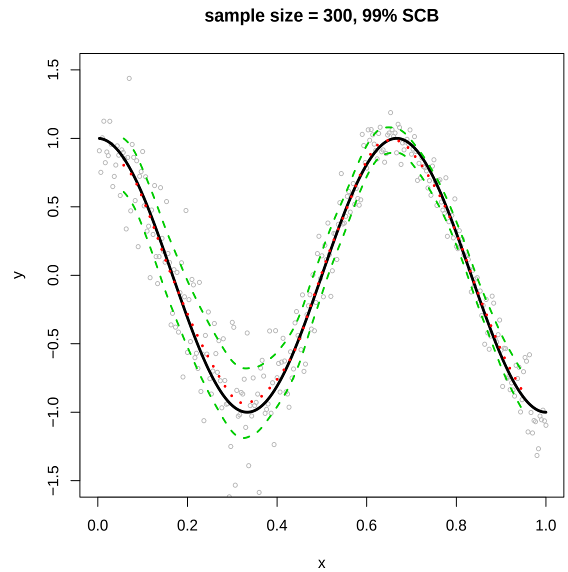


Figure 1: Boxplots of $\triangle_n = \sqrt{n}\max_j \left|\tilde{\sigma}^2_{\mathrm{K}}(x_j) - \hat{\sigma}^2_{\mathrm{SK}}(x_j)\right|$ in which $\{x_j, j = 1, 2, \ldots, n_{\text{grid}}\}$ are the points on $\tilde{\mathcal{I}}_n$ with $n_{\text{grid}} = 400$ over 2000 replications with $\varepsilon \sim N(0, 1)$: (a) Case 1; (b) Case 2; (c) Case 3; (d) Case 4.



(a)



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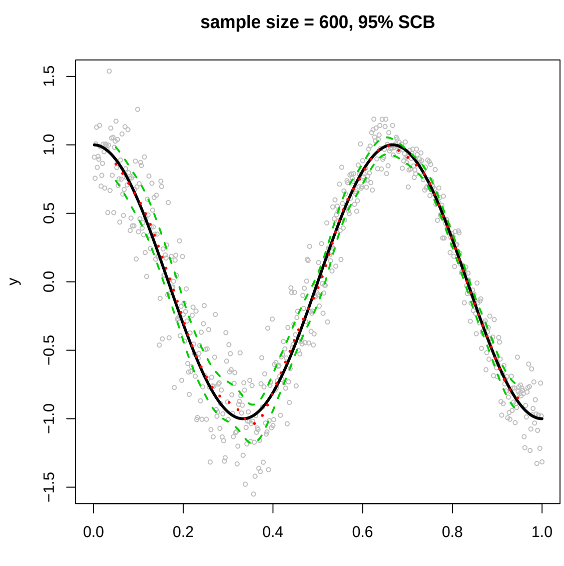

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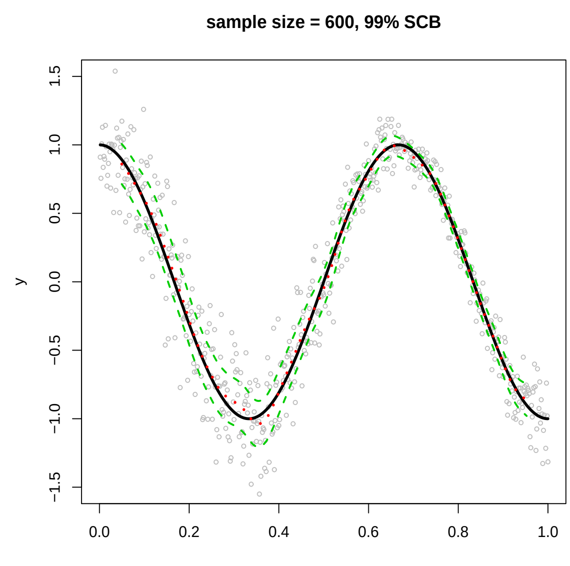


(d)

Figure 2: Plots of SCB (dashed) for $m(x)$ (solid) in Case 1 with $\varepsilon \sim N(0,1)$ which is computed according to (4.2) and the estimator $\hat{m}(x)$ (dotted).

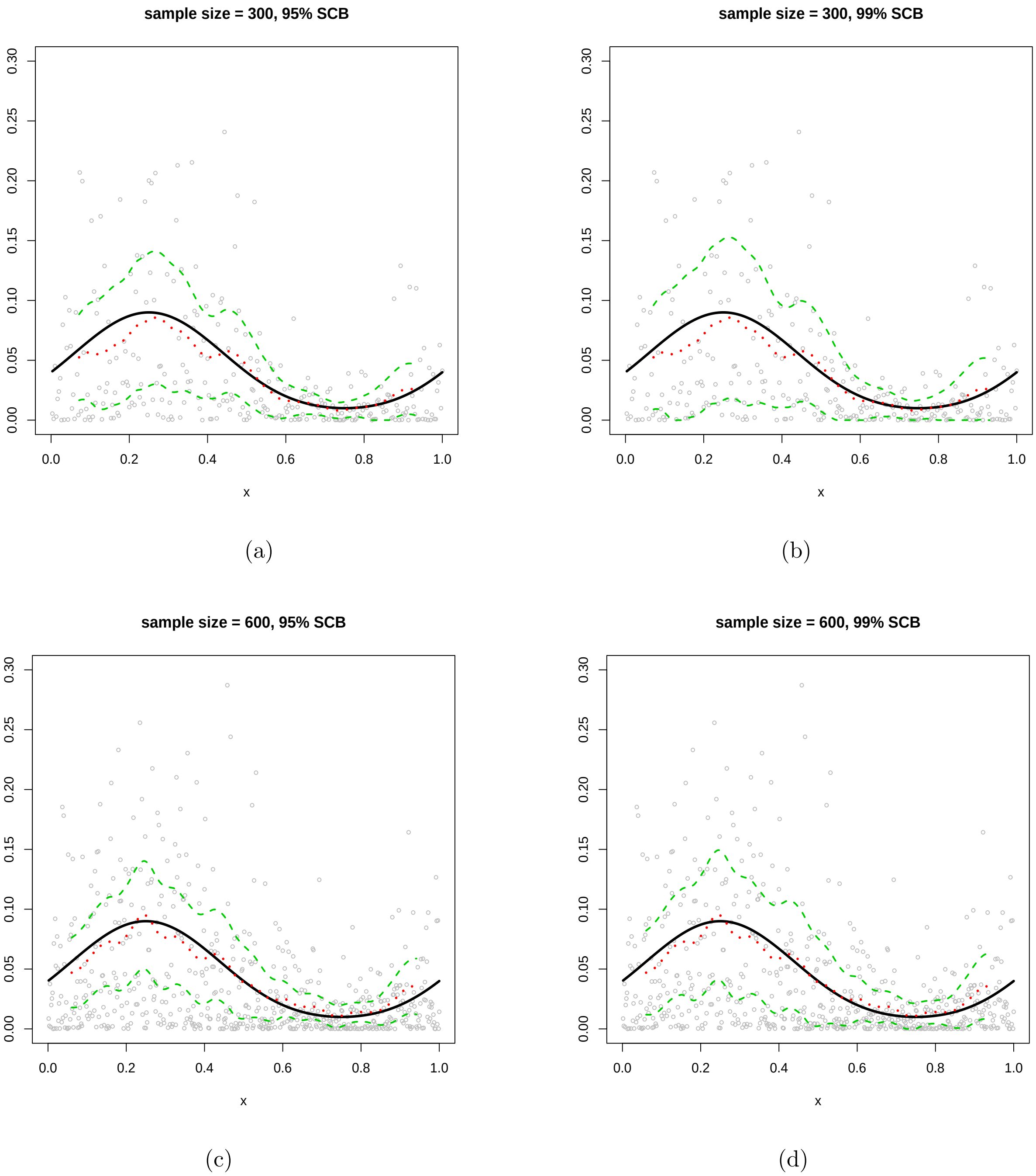
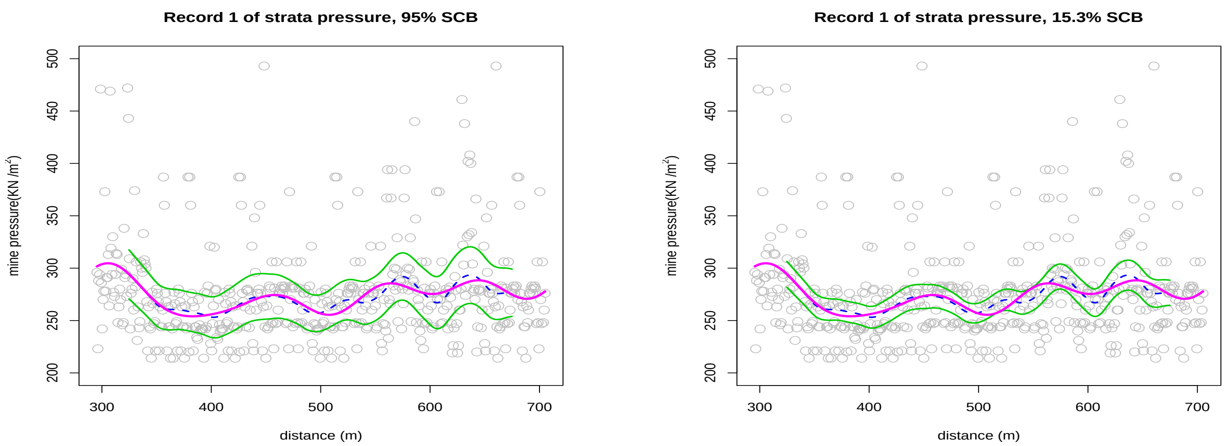
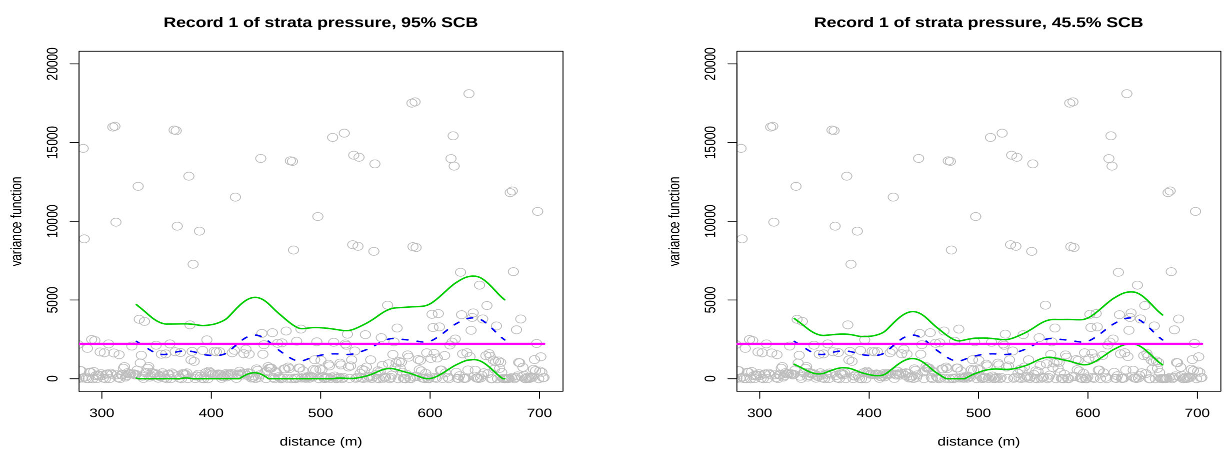


Figure 3: Plots of SCB (dashed) for $\sigma^2(x)$ (solid) in Case 1 with $\varepsilon \sim N(0,1)$ and the estimator $\hat{\sigma}^2_{\mathrm{SK}}(x)$ (dotted).



(a) (b)

Figure 4: For record 1, plots of the null hypothesis curve of $m(x) = a_0 + \sum_{k=1}^{5}\{a_k \sin(k\omega x) + b_k \cos(k\omega x)\}$ (thick solid), kernel estimator $\hat{m}(x)$ (dashed), SCB (solid) for $m(x)$ with (a) $\alpha = 0.05$ and (b) $\alpha = 0.847$.



(a) (b)

Figure 5: For record 1, plots of the null hypothesis curve of $\hat{\sigma}^2 = n^{-1}\sum_{i=1}^{n}\hat{e}_i^2$ (thick solid), SCB (solid) for $\sigma^2(x)$ and the spline-kernel estimator $\hat{\sigma}^2_{\mathrm{SK}}(x)$ (dashed) with (a) $\alpha = 0.05$ and (b) 0.545.