Supplementary Materials for "Oracally efficient estimation for dense functional data with holiday effects"

Li Cai[*1], Lisha Li[*2] , Simin Huang[2], Liang Ma[2] and Lijian Yang[2]

[1]*Zhejiang Gongshang University and* [2]*Tsinghua University*

This supplement contains all the proofs for the main results. We use c and C to denote any positive constants in the generic sense and use $\mathcal{O}(1)$ to mean 'bounded for any fixed $x \in [0,1]$' and $\mathcal{O}_{a.s.}(1)[o_{a.s.}(1)]$ to mean 'bounded [tends to 0] almost surely for any fixed $x \in [0,1]$'.

S.1 Preliminaries

In this section, we will introduce the following lemmas needed in the proofs of the main results.

Lemma S.1. *(Komlós, Major, and Tusnády, 1976, Theorem 4) Suppose* $\xi_i, 1 \leq i < \infty$*, are i.i.d. r.v.'s with* $\mathrm{E}\,\xi_1 = 0, \mathrm{E}\,\xi_1^2 = 1$*. Let* $H(x) > 0, x > 0$ *be a monotone increasing and continuous function such that for constants* $\delta > 0, x_0 > 0$*,* $x^{-3-\delta}H(x)$ *is monotone increasing for* $x > x_0$*, and* $x^{-1}\log H(x)$ *is monotone decreasing for* $x > x_0$*. Define* K_n *by the equation* $H(K_n) = n$*. If* $\mathrm{E}\,H(|\xi_1|) < \infty$*, then there exist constants* $C_1, C_2, a > 0$ *depending only on the distribution of* ξ_1 *and a sequence* $\{Z_i\}_{i=1}^n$ *of i.i.d. r.v.'s with standard normal distribution such that for any* t*,* $t > K_n, t^2/\log H(t) < C_1 n$*,*

$$P\left\{\max_{1\leq l\leq n}|S_l - W_l| > t\right\} \leq C_2 n \left\{H(at)\right\}^{-1},$$

where $S_l = \sum_{i=1}^l \xi_i$ *and* $W_l = \sum_{i=1}^l Z_i$*.*

Lemma S.2. *Assumption (A4) holds under Assumption (A4').*

Proof. Under Assumption (A4') that, $\mathrm{E}\,|\xi_{ik}|^{\eta_1+2} < \infty, \mathrm{E}\,|\varepsilon_{ij}|^{\eta_2+2} < +\infty, \eta_1 > 2, \eta_2 > 2+2\theta$, thus there exists some $\rho \in (0, 1/2)$ such that $\eta_1 > 2/\rho - 2, \eta_2 > (2+\theta)/\rho - 2$.

Let $H(x) = x^{\eta_1+2}$ which is increasing for $x > 0$. Set $x_0 = 3$ and then for $x > x_0$, $0 < \delta < 1, x^{-3-\delta}H(x)$ is monotone increasing, $x^{-1}\log H(x)$ is monotone decreasing, and hence all conditions of Lemma S.1 are met. Let $K_n = n^{1/(\eta_1+2)}, \gamma = (\eta_1+2)\rho - 1 > 1, t = C_1 n^\rho$, then one obtains that for every $1 \le k \le \infty$, there exist $C_1, C_2, a > 0$ depending only on the distribution of ξ_{ik} and standard normal variables $Z_{ik,\xi}$, such that

$$\max_{1\le k\le\infty} P\left\{\max_{1\le t\le n}\left|\sum_{i=1}^{t}\xi_{ik} - \sum_{i=1}^{t} Z_{ik,\xi}\right| > C_1 n^\rho\right\} < C_2 n^{-\gamma_1}.$$

Similarly, if one sets $H(x) = x^{\eta_2+2}$, there exist $C_1, C_2, a > 0$ depending only on the distribution of ε_{ij} and standard normal variables $Z_{ij,\varepsilon}$, such that for $t = C_1 n^\rho, \gamma_2 = (\eta_2+2)\rho - 1 - \theta > 1$,

$$\max_{1\le j\le N} P\left\{\max_{1\le t\le n}\left|\sum_{i=1}^{t}\varepsilon_{ij} - \sum_{i=1}^{t} Z_{ij,\varepsilon}\right| > C_1 n^\rho\right\} < C_2 n^{-\gamma_2-\theta}.$$

Assumption (A2) that $N = \mathcal{O}\left(n^\theta\right)$ implies that

$$P\left\{\max_{1\le j\le N}\max_{1\le t\le n}\left|\sum_{i=1}^{t}\varepsilon_{ij} - \sum_{i=1}^{t} Z_{ij,\varepsilon}\right| > C_1 n^\rho\right\} < C_2 n^{-\gamma_2}.$$

The proof is complete.

For any $t \times t$ symmetric matrix $\mathbf{A}$, let $\|\mathbf{A}\|_r = \max_{\boldsymbol{\zeta}\in\mathbb{R}^t, \boldsymbol{\zeta}\neq\mathbf{0}} \|\mathbf{A}\boldsymbol{\zeta}\|_r \|\boldsymbol{\zeta}\|_r^{-1}$ be its L_r norm. Theoretical and empirical inner products of two functions g_1, g_2 are defined as

$$\langle g_1, g_2\rangle = \int_0^1 g_1(x)\, g_2(x)\, dx, \langle g_1, g_2\rangle_{2,N} = N^{-1}\sum\nolimits_{j=1}^{N} g_1(j/N)\, g_2(j/N),$$

$$\langle g_1, g_2\rangle_{2,N\_\mathcal{V}} = N^{-1}\sum\nolimits_{j\in\bar{\mathcal{J}}_\mathcal{V}} g_1(j/N)\, g_2(j/N).$$

The corresponding norms are $\|g_1\|_2^2 = \langle g_1, g_1\rangle$, $\|g_1\|_{2,N}^2 = \langle g_1, g_1\rangle_{2,N}$, $\|g_1\|_{2,N\_\mathcal{V}}^2 = \langle g_1, g_1\rangle_{2,N\_\mathcal{V}}$. Corresponding inner product matrices of $\{b_{J,p}\}_{J=1-p}^{N_m}$ are

$$V_p = \left(\langle b_{J,p}, b_{J',p}\rangle\right)_{J=1-p}^{N_m}, \hat{V}_p = \left(\langle b_{J,p}, b_{J',p}\rangle_{2,N}\right)_{J=1-p}^{N_m}, \tag{S.1}$$

$$\hat{V}_{p\_\mathcal{V}} = \left(\langle b_{J,p}, b_{J',p}\rangle_{2,N\_\mathcal{V}}\right)_{J=1-p}^{N_m}. \tag{S.2}$$

Clearly, one has $\mathbf{B}_{\_\mathcal{V}}^T \mathbf{B}_{\_\mathcal{V}} = N\hat{V}_{p\_\mathcal{V}}$.

Lemma A.1 of Cao et al. (2016) implies the following lemma.

Lemma S.3. *For any positive integer* p, *there exists a constant* $M_p > 0$ *depending only on* p, *such that* $\left\| V_p^{-1} \right\|_\infty \leq M_p N_m$.

The theoretical and empirical inner product matrices in (S.1) and (S.2) are related in the following

Lemma S.4. *Under Assumption (A2), as* $N \to \infty$, $\left\| \hat{V}_{p\_\mathcal{V}} - V_p \right\|_\infty = \mathcal{O}\left(N^{-1}\right)$, *and for large enough* N, $\left\| \hat{V}_{p\_\mathcal{V}}^{-1} \right\|_\infty \leq 2M_p N_m$.

Proof. According to definitions in (S.1) and (S.2), matrices $\hat{V}_{p\_\mathcal{V}}$ and $\hat{V}_p$ each has at most $2p-1$ nonzero elements in each row and the boundedness of B-spline basis $\left\{b_{J,p}\left(x\right)\right\}_{J=1-p}^{N_m}$

$$\begin{aligned}
&\left\| \hat{V}_{p_\mathcal{V}} - \hat{V}_p \right\|_\infty \\
&\leq \left(2p-1\right) \max_{1-p \leq J,J' \leq N_m} \left| \left\langle b_{J,p}\left(x\right), b_{J',p}\left(x\right) \right\rangle_{2,N_\mathcal{V}} - \left\langle b_{J,p}\left(x\right), b_{J',p}\left(x\right) \right\rangle_{2,N} \right| \\
&= \left(2p-1\right) \max_{1-p \leq J,J' \leq N_m} N^{-1} \left| \sum_{j \in \bar{\mathcal{I}}_\mathcal{V}} b_{J,p}\left(j/N\right) b_{J',p}\left(j/N\right) - \sum_{j=1}^{N} b_{J,p}\left(j/N\right) b_{J',p}\left(j/N\right) \right| \\
&= \left(2p-1\right) \max_{1-p \leq J,J' \leq N_m} N^{-1} \left| \sum_{\nu=1}^{\mathcal{V}} b_{J,p}\left(j_\nu/N\right) b_{J',p}\left(j_\nu/N\right) \right| \\
&= \left(2p-1\right) \mathcal{V} N^{-1}.
\end{aligned}$$

As Lemma A.3 of Cao et al. (2016) provides that $\left\| V_p - \hat{V}_p \right\|_\infty = \mathcal{O}\left(N^{-1}\right)$, therefore

$$\left\| \hat{V}_{p\_\mathcal{V}} - V_p \right\|_\infty \leq \left\| \hat{V}_{p\_\mathcal{V}} - \hat{V}_p \right\|_\infty + \left\| \hat{V}_p - V_p \right\|_\infty = \mathcal{O}\left(N^{-1}\right).$$

Next Lemma S.3 implies that for any $\gamma \in \mathbb{R}^{N_m+p}$, $\left\| V_p^{-1}\gamma \right\|_\infty \leq M_p N_m \left\|\gamma\right\|_\infty$, so $\left\| V_p\gamma \right\|_\infty \geq M_p^{-1} N_m^{-1} \left\|\gamma\right\|_\infty$. By Assumption (A3), $N^{-1} = o\left(N_m\right)$, thus if N is large enough, for any $\gamma \in \mathbb{R}^{N_m+p}$,

$$\begin{aligned}
\left\| \hat{V}_{p_\mathcal{V}}\gamma \right\|_\infty \geq \left\| V_p\gamma \right\|_\infty - \left\| \left(V_p - \hat{V}_{p_\mathcal{V}}\right)\gamma \right\|_\infty &\geq M_p^{-1} N_m^{-1} \left\|\gamma\right\|_\infty - \mathcal{O}\left(N^{-1}\right) \left\|\gamma\right\|_\infty \\
&= 2^{-1} M_p^{-1} N_m^{-1} \left\|\gamma\right\|_\infty.
\end{aligned}$$

Therefore $\left\| \hat{V}_{p\_\mathcal{V}}^{-1} \right\|_\infty \leq 2M_p N_m$. The proof is complete.

Lemma S.5. *There is an absolute constant $C_{p-1,\mu} > 0$ such that for every $\phi \in C^{p-1,\mu}[0,1]$, there exists a function $g \in \mathcal{H}_{N_m}^{(p-2)}$ such that*

$$\|g-\phi\|_\infty \le C_{p-1,\mu}\left\|\phi^{(p-1)}\right\|_{0,\mu} N_m^{-(\mu+p-1)}.$$

Proof. The lemma follows directly from Theorem 6, p.149 of de Boor (1978).

Lemma S.6. *There exists $c_p \in (0,\infty)$ such that $\left\|\tilde{\phi}\right\|_\infty \le c_p \|\phi\|_\infty$, for any $\phi \in C[0,1]$. Furthermore, if $\phi \in C^{p-1,\mu}[0,1]$ for some $\mu \in (0,1]$, then for $\tilde{C}_{p-1,\mu} = (c_p+1)\,C_{p-1,\mu}$,*

$$\left\|\tilde{\phi}-\phi\right\|_\infty \le \tilde{C}_{p-1,\mu}\left\|\phi^{(p-1)}\right\|_{0,\mu} N_m^{-(\mu+p-1)}.$$

Proof. Note that for any $x \in [0,1]$ at most $(p+1)$ of the numbers $b_{1-p,p}(x), ..., b_{N_m,p}(x)$ are between 0 and 1, others being 0, therefore

$$\begin{aligned}
\left\|\tilde{\phi}\right\|_\infty &\le (p+1)\left\|\left(\mathbf{B}_{\text{-}\mathcal{V}}^T\mathbf{B}_{\text{-}\mathcal{V}}\right)^{-1}\mathbf{B}_{\text{-}\mathcal{V}}^T\boldsymbol{\phi}_{\text{-}\mathcal{V}}\right\|_\infty = (p+1)\left\|\hat{V}_{p\text{-}\mathcal{V}}^{-1}\mathbf{B}_{\text{-}\mathcal{V}}^T\boldsymbol{\phi}_{\text{-}\mathcal{V}}N^{-1}\right\|_\infty \\
&\le (p+1)\left\|\hat{V}_{p\text{-}\mathcal{V}}^{-1}\right\|_\infty \left\|\mathbf{B}_{\text{-}\mathcal{V}}^T\boldsymbol{\phi}_{\text{-}\mathcal{V}}N^{-1}\right\|_\infty \\
&\le (p+1)\times 2M_pN_m\left\|\mathbf{B}_{\text{-}\mathcal{V}}^T\mathbf{I}_{N\text{-}\mathcal{V}}N^{-1}\right\|_\infty \|\phi\|_\infty,
\end{aligned}$$

in which $\mathbf{I}_{N\text{-}\mathcal{V}} = (1,...,1)^T$. Obviously, $\left\|\mathbf{B}_{\text{-}\mathcal{V}}^T\mathbf{I}_{N\text{-}\mathcal{V}}N^{-1}\right\|_\infty \le CN_m^{-1}$ for some constant $C>0$, hence $\left\|\tilde{\phi}\right\|_\infty \le (p+1)\times 2M_pC\|\phi\|_\infty = c_p\|\phi\|_\infty$.

Now if $\phi \in C^{p-1,\mu}[0,1]$ for some $\mu \in (0,1]$, according to Lemma S.5, there exists a function $g \in \mathcal{H}_{N_m}^{(p-2)}$ such that $\|g-\phi\|_\infty \le C_{p-1,\mu}\left\|\phi^{(p-1)}\right\|_{0,\mu} N_m^{-(\mu+p-1)}$. Hence,

$$\begin{aligned}
\left\|\tilde{\phi}-\phi\right\|_\infty &\le \left\|\tilde{\phi}-g\right\|_\infty + \|g-\phi\|_\infty \\
&\le (c_p+1)\|g-\phi\|_\infty \le (c_p+1)\,C_{p-1,\mu}\left\|\phi^{(p-1)}\right\|_{0,\mu} N_m^{-(\mu+p-1)},
\end{aligned}$$

which completes the proof of lemma.

The following strong approximation properties are proved in Lemma A.5 of Cao et al. (2012).

Lemma S.7. *Under Assumption (A4), for $n \ge 1$,*

$$\max_{1\le j\le N}\left|\bar{\varepsilon}_{\cdot j} - \bar{Z}_{\cdot j,\varepsilon}\right| = \mathcal{O}_{a.s.}\left(n^{\rho-1}\right),$$

and

$$\max_{1\leq k\leq\infty} \mathrm{E}\left|\bar{\xi}_{\cdot k}\right| \leq n^{-1/2}\left(\frac{2}{\pi}\right)^{1/2} + C_0 n^{\rho-1}, \tag{S.3}$$

where $\bar{\varepsilon}_{\cdot j} = n^{-1}\sum_{i=1}^{n}\varepsilon_{ij}, \bar{\xi}_{\cdot j} = n^{-1}\sum_{i=1}^{n}\xi_{ij}, \bar{Z}_{\cdot j,\varepsilon} = n^{-1}\sum_{i=1}^{n} Z_{ik,\varepsilon}$.

S.2 Proofs of Propositions and Theorems

Proof of Proposition 2. For $\tilde{m}(x)$, $\tilde{\phi}_k(x)$ in (10) and (11), Lemma **S.6** and Assumptions (A1), (A3) imply that

$$\left\|\tilde{m}(x) - m(x)\right\|_\infty \leq \tilde{C}_{p-1,1}\left\|m^{(p-1)}\right\|_{0,1} N_m^{-p},$$
$$\left\|\tilde{\phi}_k(x) - \phi(x)\right\|_\infty \leq \tilde{C}_{0,\mu}\left\|\phi(x)\right\|_{0,\mu} N_m^{-\mu}.$$

Assumptions (A2) and (A3) imply $N_m^{-p} n^{1/2} = o(1)$ and $C(x,x) > C$, hence

$$\sup_{x\in[0,1]} C^{-1/2}(x,x)\left|\tilde{m}(x) - m(x)\right| = o_p\left(n^{-1/2}\right), \tag{S.1}$$

and according to (S.3) in Lemma S.7, one has that

$$\begin{aligned}
&\mathrm{E}\sup_{x\in[0,1]}\left|C^{-1/2}(x,x)\sum\nolimits_{k=1}^{\infty}\bar{\xi}_{\cdot k}\left\{\tilde{\phi}_k(x) - \phi_k(x)\right\}\right| \\
&\leq C\left\{\sum\nolimits_{k=1}^{\kappa_n}\mathrm{E}\left|\bar{\xi}_{\cdot k}\right|\left\|\phi_k(x)\right\|_{0,\mu} N_m^{-\mu} + \sum\nolimits_{k=\kappa_n+1}^{\infty}\mathrm{E}\left|\bar{\xi}_{\cdot k}\right|\left\|\phi_k(x)\right\|_\infty\right\} \\
&= o\left(n^{-1/2}\right).
\end{aligned} \tag{S.2}$$

Notice next that

$$\begin{aligned}
&\left|n^{-1}\sum\nolimits_{i=1}^{n}\left\{\tilde{\xi}_i(x) - \bar{m}(x)\right\}C(x,x)^{-1/2}\right| \\
&\quad = C(x,x)^{-1/2}\left|\tilde{m}(x) - m(x) + \sum\nolimits_{k=1}^{\infty}\bar{\xi}_{\cdot k}\left\{\tilde{\phi}_k(x) - \phi_k(x)\right\}\right|.
\end{aligned} \tag{S.3}$$

Therefore (S.1), (S.2) and (S.3) conclude the proof of proposition.

Proof of Proposition 3. Using decomposition (10), for $\mathbf{E}_{\text{-}\mathcal{V}} = \left(\sigma(j/N)\bar{\varepsilon}_{\cdot j}\ \ , j \in \bar{\mathcal{I}}_{\mathcal{V}}\right)^T$

$$n^{-1}\sum\nolimits_{i=1}^{n}\tilde{\varepsilon}_i(x)C(x,x)^{-1/2} \ = \ C(x,x)^{-1/2}\mathbf{b}(x)\left(\mathbf{B}_{\text{-}\mathcal{V}}^T\mathbf{B}_{\text{-}\mathcal{V}}\right)^{-1}\mathbf{B}_{\text{-}\mathcal{V}}^T\mathbf{E}_{\text{-}\mathcal{V}}.$$

Denote $\tilde{e}_p(x) = \mathbf{b}(x)\left(\mathbf{B}_{\text{-}\mathcal{V}}^T \mathbf{B}_{\text{-}\mathcal{V}}\right)^{-1} \mathbf{B}_{\text{-}\mathcal{V}}^T \mathbf{E}_{\text{-}\mathcal{V}}$, and $\tilde{Z}_{p,\varepsilon}(x) = \mathbf{b}(x)\left(\mathbf{B}_{\text{-}\mathcal{V}}^T \mathbf{B}_{\text{-}\mathcal{V}}\right)^{-1} \times \mathbf{B}_{\text{-}\mathcal{V}}^T \mathbf{Z}_{\text{-}\mathcal{V}}$ with $\mathbf{Z}_{\text{-}\mathcal{V}} = \left(\sigma(j/N)\bar{Z}_{\cdot j,\varepsilon}, j \in \bar{\mathcal{I}}_{\mathcal{V}}\right)^T$. Lemma S.4 entails that for large enough N,

$$\begin{aligned} &\sup_{x\in[0,1]} \left|\left\{\tilde{Z}_{p,\varepsilon}(x) - \tilde{e}_p(x)\right\} C^{-1/2}(x,x)\right| \\ &= \sup_{x\in[0,1]} \left|C^{-1/2}(x,x)\,\mathbf{b}(x)\,\hat{V}_{p\text{-}\mathcal{V}}^{-1} N^{-1} \mathbf{B}_{\text{-}\mathcal{V}}^T \left(\mathbf{Z}_{\text{-}\mathcal{V}} - \mathbf{E}_{\text{-}\mathcal{V}}\right)\right| \\ &\leq C(p+1)\left\|\hat{V}_{p\text{-}\mathcal{V}}^{-1}\right\|_\infty \left\|N^{-1}\mathbf{B}_{\text{-}\mathcal{V}}^T\left(\mathbf{Z}_{\text{-}\mathcal{V}} - \mathbf{E}_{\text{-}\mathcal{V}}\right)\right\|_\infty \\ &\leq C(p+1)\,2M_p N_m \left\|N^{-1}\mathbf{B}_{\text{-}\mathcal{V}}^T\left(\mathbf{Z}_{\text{-}\mathcal{V}} - \mathbf{E}_{\text{-}\mathcal{V}}\right)\right\|_\infty. \end{aligned} \tag{S.4}$$

Note that Lemma **S.7** implies that $\|\mathbf{Z}_{\text{-}\mathcal{V}} - \mathbf{E}_{\text{-}\mathcal{V}}\|_\infty = \mathcal{O}_{a.s.}(n^{\rho-1})$, thus

$$\begin{aligned} \left\|N^{-1}\mathbf{B}_{\text{-}\mathcal{V}}^T\left(\mathbf{Z}_{\text{-}\mathcal{V}} - \mathbf{E}_{\text{-}\mathcal{V}}\right)\right\|_\infty &\leq \|(\mathbf{Z}_{\text{-}\mathcal{V}} - \mathbf{E}_{\text{-}\mathcal{V}})\|_\infty \max_{1-p\leq J\leq N_m} \langle B_{J,p}, 1\rangle_{2,N\text{-}\mathcal{V}} \\ &\leq \mathcal{O}_{a.s.}\left(n^{\rho-1}\right)\mathcal{O}\left(N_m^{-1}\right) \\ &= \mathcal{O}_{a.s.}\left(n^{\rho-1}N_m^{-1}\right), \end{aligned}$$

which together with (**S.4**) and Assumption (A4) implies

$$\sup_{x\in[0,1]} \left|\left\{\tilde{Z}_{p,\varepsilon}(x) - \tilde{e}_p(x)\right\} C^{-1/2}(x,x)\right| = \mathcal{O}_{a.s.}\left(n^{\rho-1}\right), \rho < 1/2. \tag{S.5}$$

Notice that the $(N_m + p)$ random vector $\hat{V}_{p\text{-}\mathcal{V}}^{-1} N^{-1}\mathbf{B}_{\text{-}\mathcal{V}}^T \mathbf{Z}_{\text{-}\mathcal{V}}$ follows normal distribution with mean 0 and covariance matrix $\hat{V}_{p\text{-}\mathcal{V}}^{-1} N^{-1}\mathbf{B}_{\text{-}\mathcal{V}}^T \operatorname{var}(\mathbf{Z}_{\text{-}\mathcal{V}}) \times \left(\hat{V}_{p\text{-}\mathcal{V}}^{-1} N^{-1}\mathbf{B}_{\text{-}\mathcal{V}}^T\right)^T$, whose largest eigenvalue is bounded by

$$\max_{x\in[0,1]} N^{-1}n^{-1}\sigma^2(x)\left\|\hat{V}_{p\text{-}\mathcal{V}}^{-1}\hat{V}_{p\text{-}\mathcal{V}}\hat{V}_{p\text{-}\mathcal{V}}^{-1}\right\|_\infty \leq CN^{-1}n^{-1}N_m.$$

Therefore by the tail property of the normal distribution and the Borel-Cantelli lemma

$$\left\|\hat{V}_{p\text{-}\mathcal{V}}^{-1} N^{-1}\mathbf{B}_{\text{-}\mathcal{V}}^T \mathbf{Z}_{\text{-}\mathcal{V}}\right\|_\infty = \mathcal{O}_{a.s.}\left(N^{-1/2}n^{-1/2}N_m^{1/2}\log^{1/2}(N_m+p)\right),$$

since $\log(N_m + p) = \mathcal{O}(\log n)$ by Assumption (A2),

$$\left\|\tilde{Z}_{p,\varepsilon}(x)\right\|_\infty = \mathcal{O}_{a.s.}\left(N^{-1/2}n^{-1/2}N_m^{1/2}\log^{1/2} n\right). \tag{S.6}$$

Putting Assumption (A2), (S.4), (S.5) and (S.6) together, one has proved the proposition.

Proof of Theorem 1. According to (2), (3), (4), (5) and (10)

$$\sup_{x\in[0,1]} \left|\hat{m}(x) - \bar{m}(x)\right| = \sup_{x\in[0,1]} \left|n^{-1}\sum\nolimits_{i=1}^{n}\left\{\tilde{\xi}_i(x) - \bar{m}(x)\right\} + n^{-1}\sum\nolimits_{i=1}^{n}\tilde{\varepsilon}_i(x)\right|$$

and

$$\begin{aligned}
\left\|\hat{\boldsymbol{\beta}} - \tilde{\boldsymbol{\beta}}\right\|_\infty &= \max_{1\leq\nu\leq\mathcal{V}}\left|\hat{\beta}_\nu - \tilde{\beta}_\nu\right| = \max_{1\leq\nu\leq\mathcal{V}}\left|n^{-1}\sum\nolimits_{i=1}^{n}\hat{e}_{ij_\nu} - n^{-1}\sum\nolimits_{i=1}^{n}e_{ij_\nu}\right| \\
&= n^{-1}\max_{1\leq\nu\leq\mathcal{V}}\left|\sum\nolimits_{i=1}^{n}\left\{Y_{ij_\nu} - \hat{\xi}_i(j_\nu/N)\right\} - \sum\nolimits_{i=1}^{n}\left\{Y_{ij_\nu} - \xi_i(j_\nu/N)\right\}\right| \\
&= n^{-1}\max_{1\leq\nu\leq\mathcal{V}}\left|\sum\nolimits_{i=1}^{n}\left\{\tilde{\xi}_i(j_\nu/N) + \tilde{\varepsilon}_i(j_\nu/N)\right\} - \sum\nolimits_{i=1}^{n}\xi_i(j_\nu/N)\right| \\
&= \max_{1\leq\nu\leq\mathcal{V}}\left|n^{-1}\sum\nolimits_{i=1}^{n}\left\{\tilde{\xi}_i(j_\nu/N) - \bar{m}(j_\nu/N)\right\} + n^{-1}\sum\nolimits_{i=1}^{n}\tilde{\varepsilon}_i(j_\nu/N)\right|.
\end{aligned}$$

Thus Assumption (A3), Propositions 2 and 3 conclude Theorem 1.

Proof of Theorem 2. Note that for $1 \leq \nu \leq \mathcal{V}$,

$$\tilde{\beta}_\nu - \beta_\nu = n^{-1}\sum\nolimits_{i=1}^{n}\sigma(j_\nu/N)\,\varepsilon_{ij_\nu},$$

so for $\Sigma = \mathrm{diag}\left(\sigma^2(j_1/N), ..., \sigma^2(j_{\mathcal{V}}/N)\right)$, identity matrix $\mathbf{I}_{\mathcal{V}}$, Central Limit Theorem implies that

$$n^{1/2}\Sigma^{-1/2}\left(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) \to_D \mathcal{N}\left(\mathbf{0}, \mathbf{I}_{\mathcal{V}}\right),$$

which together with (8) in Theorem 1, imply Theorem 2.