ORACLE-EFFICIENT CONFIDENCE ENVELOPES FOR COVARIANCE FUNCTIONS IN DENSE FUNCTIONAL DATA

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Supplementary Material

This note contains proofs for Lemmas 3, 4 and 6, Propositions 1 and 2.

S1 Proofs for some technical lemmas

In this section, we provide proofs for Lemmas 3, 4 and 6.

S1.1 Proof of Lemma 3.

Note that

$$\begin{aligned}
&\hat{\mathbf{V}}_{p_2,2} \\
= &\left\{ N^{-2} \sum_{1\le j\ne j'\le N} B_{JJ',p_2}(j/N, j'/N)\, B_{J''J''',p_2}(j/N, j'/N) \right\}_{J,J',J'',J'''=1-p_2}^{N_{s_2}} \\
= &\left\{ N^{-2} \left[\sum_{j=1}^{N} B_{JJ'',p_2}(j/N, j/N) \right] \left[\sum_{j=1}^{N} B_{J'J''',p_2}(j/N, j/N) \right] \right\}_{J,J',J'',J'''=1-p_2}^{N_{s_2}} \\
&- \left\{ N^{-2} \sum_{j=1}^{N} B_{JJ',p_2}(j/N, j/N)\, B_{J''J''',p_2}(j/N, j/N) \right\}_{J,J',J'',J'''=1-p_2}^{N_{s_2}} \\
= &\hat{\mathbf{V}}_{p_2} \otimes \hat{\mathbf{V}}_{p_2} \\
&- \left\{ N^{-2} \sum_{j=1}^{N} B_{JJ',p_2}(j/N, j/N)\, B_{J''J''',p_2}(j/N, j/N) \right\}_{J,J',J'',J'''=1-p_2}^{N_{s_2}}.
\end{aligned}$$

Note that the entries in the matrix $\hat{\mathbf{V}}_{p_2,2} - \hat{\mathbf{V}}_{p_2} \otimes \hat{\mathbf{V}}_{p_2}$ are zero when the maximum absolute difference between any two of the indices (J, J''), (J', J''') is greater than p;

otherwise

$$
\begin{aligned}
& N^{-2} \sum_{j=1}^{N} B_{JJ',p_2}\left(j/N, j/N\right) B_{J''J''',p_2}\left(j/N, j/N\right) \\
& = N^{-1}\left[\int_0^1 B_{JJ',p_2}\left(x,x\right) B_{J''J''',p_2}\left(x,x\right) dx + O(N^{-1}h_{s_2}^{-1})\right] \\
& = O\left(N^{-1}h_{s_2} + N^{-2}h_{s_2}^{-1}\right).
\end{aligned}
$$

Hence, $\left\|\hat{\mathbf{V}}_{p_2,2} - \hat{\mathbf{V}}_{p_2} \otimes \hat{\mathbf{V}}_{p_2}\right\|_\infty = O\left(N^{-1}h_{s_2} + N^{-2}h_{s_2}^{-1}\right)$. Since

$$
\begin{aligned}
& \left\|\mathbf{V}_{p_2} \otimes \mathbf{V}_{p_2} - \hat{\mathbf{V}}_{p_2} \otimes \hat{\mathbf{V}}_{p_2}\right\|_\infty \\
= & \max_{1-p_2 \le J',J''' \le N_{s_2}} \sum_{J,J''=1-p_2}^{N_{s_2}} \left| N^{-2} \sum_{j,j'=1}^{N} \left[B_{JJ',p_2}\left(j/N, j'/N\right) B_{J''J''',p_2}\left(j/N, j'/N\right)\right] \right. \\
& \left. - \int_0^1 \int_0^1 B_{JJ',p_2}\left(x,x'\right) B_{J''J''',p_2}\left(x,x'\right) dx dx' \right| \\
\le & \max_{1-p_2 \le J',J''' \le N_{s_2}} \sum_{J,J''=1-p_2}^{N_{s_2}} \sum_{j,j'=1}^{N} \int_{(j'-1)/N}^{j'/N} \int_{(j-1)/N}^{j/N} \left| B_{JJ',p_2}\left(j/N, j'/N\right) \right. \\
& \left. \times B_{J''J''',p_2}\left(j/N, j'/N\right) - B_{JJ',p_2}\left(x,x'\right) B_{J''J''',p_2}\left(x,x'\right) \right| dx dx' \\
\le & C h_{s_2}^{-2} \left(N h_{s_2}\right)^2 \times N^{-2} \times N^{-2} h_{s_2}^{-2} = C N^{-2} h_{s_2}^{-2},
\end{aligned}
$$

applying Assumption (A3) one has $\left\|\mathbf{V}_{p_2} \otimes \mathbf{V}_{p_2} - \hat{\mathbf{V}}_{p_2,2}\right\|_\infty = o\left(N^{-1}\right)$.

According to Lemma 2, for any $\left(N_{s_2} + p_2\right)^2$ vector $\boldsymbol{\tau}$, one has $\left\|(\mathbf{V}_{p_2} \otimes \mathbf{V}_{p_2})^{-1}\boldsymbol{\tau}\right\|_\infty \le h_{s_2}^{-2} \left\|\boldsymbol{\tau}\right\|_\infty$. Hence, $\left\|(\mathbf{V}_{p_2} \otimes \mathbf{V}_{p_2})\boldsymbol{\tau}\right\|_\infty \ge h_{s_2}^2 \left\|\boldsymbol{\tau}\right\|_\infty$. Note that

$$
\begin{aligned}
\left\|\hat{\mathbf{V}}_{p_2,2}\boldsymbol{\tau}\right\|_\infty & \ge \left\|(\mathbf{V}_{p_2} \otimes \mathbf{V}_{p_2})\boldsymbol{\tau}\right\|_\infty - \left\|(\mathbf{V}_{p_2} \otimes \mathbf{V}_{p_2})\boldsymbol{\tau} - \hat{\mathbf{V}}_{p_2,2}\boldsymbol{\tau}\right\|_\infty \\
& = O\left(h_{s_2}^2\right) \left\|\boldsymbol{\tau}\right\|_\infty.
\end{aligned}
$$

If $\boldsymbol{\tau}$ satisfies that $\left\|\hat{\mathbf{V}}_{p_2,2}^{-1}\right\|_\infty = \left\|\hat{\mathbf{V}}_{p_2,2}^{-1}\boldsymbol{\tau}\right\|_\infty \le O\left(h_{s_2}^{-2}\right) \left\|\boldsymbol{\tau}\right\|_\infty = O\left(h_{s_2}^{-2}\right)$, the lemma is proved.

S1.2 Proof of Lemma 4.

Note that for any matrix $\mathbf{A} = (a_{ij})_{i=1,j=1}^{m,n}$ and any n by 1 vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)^{\mathrm{T}}$, one has $\|\mathbf{A}\boldsymbol{\alpha}\|_\infty \le \|\mathbf{A}\|_\infty \|\boldsymbol{\alpha}\|_\infty$. It is clear that

$$\begin{aligned}\left\|N^{-2}\mathbf{X}^{\mathrm{T}}\boldsymbol{\rho}\right\|_\infty &= \left\|N^{-2}\left\{\sum_{1\le j\ne j'\le N} B_{JJ',p_2}(j/N, j'/N)\,\rho_{jj'}\right\}_{J,J'=1-p_2}^{N_{s_2}}\right\|_\infty \\ &\le \|\boldsymbol{\rho}\|_\infty \max_{1-p_2\le J,J'\le N_{s_2}} \left|N^{-2}\sum_{1\le j\ne j'\le N} B_{JJ',p_2}(j/N, j'/N)\right| \\ &\le h_{s_2}^2 \|\boldsymbol{\rho}\|_\infty .\end{aligned}$$

One also observe that

$$\left\|N^{-2}\boldsymbol{B}_{p_2}^{\mathrm{T}}(x,x')\hat{\mathbf{V}}_{p_2,2}^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{\rho}\right\|_\infty \le \left\|\boldsymbol{B}_{p_2}^{\mathrm{T}}(x,x')\right\|_\infty \left\|N^{-2}\mathbf{X}^{\mathrm{T}}\boldsymbol{\rho}\right\|_\infty \left\|\hat{\mathbf{V}}_{p_2,2}^{-1}\right\|_\infty,$$

which, together with the boundedness of spline functions and Lemma 3, leads to the desired result.

S1.3 Proof of Lemma 6.

Let $\mathcal{F}_t = \sigma(\bar{\xi}_{\cdot 11}, \bar{\xi}_{\cdot 12}, \ldots, \bar{\xi}_{\cdot 1t}, \bar{\xi}_{\cdot 22}, \ldots, \bar{\xi}_{\cdot t-1,t}, \bar{\xi}_{\cdot tt})$, then $\mathcal{F}_2 \subseteq \mathcal{F}_3 \subseteq \mathcal{F}_4 \subseteq \cdots$ is an increasing sequence of σ-fields. Define

$$S_t(x,x') = \sqrt{n}\sum_{1\le k\ne k'\le t}\bar{\xi}_{\cdot kk'}\phi_k(x)\phi_{k'}(x') + \sqrt{n}\sum_{1\le k\le t}\left(\bar{\xi}_{\cdot kk}-1\right)\phi_k(x)\phi_k(x'),$$

for $t = 2, 3, \ldots, k_n$, where $\lim_{n\to\infty} k_n = \infty$. We first show that $S_t(x,x')$ is a martingale process indexed by $[0,1]^2$, i.e. for each $(x,x') \in [0,1]^2$.

Let

$$\begin{aligned}d_t(x,x') &= S_t(x,x') - S_{t-1}(x,x') \\ &= \sqrt{n}\left\{\sum_{k=1}^{t-1}\bar{\xi}_{\cdot kt}\left(\phi_k(x)\phi_t(x') + \phi_t(x)\phi_k(x')\right) + \left(\bar{\xi}_{\cdot tt}-1\right)\phi_t(x)\phi_t(x')\right\},\end{aligned}$$

then $d_t(x,x')$ is $\mathcal{F}_t$-measurable for each $(x,x') \in [0,1]^2$. Next note that for any t,

$$\begin{aligned}& E\{d_t(x,x')|\mathcal{F}_{t-1}\} \\ =\ & \sqrt{n}E\left\{\sum_{k=1}^{t-1}\bar{\xi}_{\cdot kt}\left(\phi_k(x)\phi_t(x') + \phi_t(x)\phi_k(x')\right) + \left(\bar{\xi}_{\cdot tt}-1\right)\phi_t(x)\phi_t(x')\middle|\,\mathcal{F}_{t-1}\right\} \\ =\ & \sqrt{n}E\left[n^{-1}\sum_{i=1}^{n}\xi_{it}\left\{\sum_{k=1}^{t-1}\xi_{ik}\left(\phi_k(x)\phi_t(x') + \phi_t(x)\phi_k(x')\right)\right\}\middle|\,\mathcal{F}_{t-1}\right] \\ & + \sqrt{n}E\left\{\left(\bar{\xi}_{\cdot tt}-1\right)\phi_t(x)\phi_t(x')\middle|\,\mathcal{F}_{t-1}\right\} \\ =\ & 0.\end{aligned}$$

Hence $\{d_t(x,x'), t=2,3,\ldots\}$ is a martingale difference process indexed by $[0,1]^2$ with respect to $\{\mathcal{F}_t, t=2,3,\ldots\}$.

Note that

$$E\{d_t^2(x,x')|\mathcal{F}_{t-1}\} = v_t^{(1)}(x,x') + v_t^{(2)}(x,x') + v_t^{(3)}(x,x'), \quad \text{(S.1)}$$

where

$$\begin{aligned}
v_t^{(1)}(x,x') &= nE\left[\left.\left\{\sum_{k=1}^{t-1}\bar{\xi}_{\cdot kt}\left(\phi_k(x)\phi_t(x')+\phi_t(x)\phi_k(x')\right)\right\}^2\right|\mathcal{F}_{t-1}\right],\\
v_t^{(2)}(x,x') &= nE\left[\left\{\left(\bar{\xi}_{\cdot tt}-1\right)\phi_t(x)\phi_t(x')\right\}^2|\mathcal{F}_{t-1}\right],\\
v_t^{(3)}(x,x') &= 2nE\left[\left\{\sum_{k=1}^{t-1}\bar{\xi}_{\cdot kt}\left(\phi_k(x)\phi_t(x')+\phi_t(x)\phi_k(x')\right)\right\}\left(\bar{\xi}_{\cdot tt}-1\right)\phi_t(x)\phi_t(x')|\mathcal{F}_{t-1}\right].
\end{aligned}$$

For $v_t^{(1)}(x,x')$, one has

$$\begin{aligned}
v_t^{(1)}(x,x') &= E\left[\frac{1}{n}\sum_{i=1}^n\left\{\sum_{k=1}^{t-1}\xi_{ik}\left(\phi_k(x)\phi_t(x')+\phi_t(x)\phi_k(x')\right)\right\}^2\xi_{it}^2\Big|\,\mathcal{F}_{t-1}\right]\\
&= E\sum_{1\le k\le t-1}\left[\left.\frac{1}{n}\sum_{i=1}^n\left\{\xi_{ik}^2\xi_{it}^2\left(\phi_k(x)\phi_t(x')+\phi_t(x)\phi_k(x')\right)^2\right\}\right|\mathcal{F}_{t-1}\right]\\
&= E\xi_{1t}^2\sum_{1\le k\le t-1}\bar{\xi}_{\cdot kk}\left(\phi_k^2(x)\phi_t^2(x')+\phi_k^2(x')\phi_t^2(x)\right.\\
&\quad\left.+2\phi_k(x)\phi_k(x')\phi_t(x')\phi_t(x)\right),
\end{aligned}$$

so we have for $n\to\infty$ and $k_n\to\infty$,

$$\sum_{t=2}^{k_n}v_t^{(1)}(x,x') \to \sum_{k\neq k'}^{\infty}\phi_k^2(x)\phi_{k'}^2(x') + \sum_{k\neq k'}^{\infty}\phi_k(x)\phi_{k'}(x')\phi_k(x')\phi_{k'}(x) < \infty$$

in probability.

Next note that $v_t^{(2)}(x,x') = \left(E\xi_{1t}^4-1\right)\phi_t^2(x)\phi_t^2(x')$, so one has for any $k_n\ge 1$,

$$\sum_{t=2}^{k_n}v_t^{(2)}(x,x') \to \sum_{k=1}^{\infty}\left(E\xi_{1k}^4-1\right)\phi_k^2(x)\phi_k^2(x') < \infty.$$

For $v_t^{(3)}(x,x')$, one has

$$\begin{aligned}
&v_t^{(3)}(x,x')\\
=\;& 2nE\left[\left\{\sum_{k=1}^{t-1}\frac{1}{n}\sum_{j=1}^n\xi_{jk}\xi_{jt}\left(\phi_k(x)\phi_t(x')+\phi_t(x)\phi_k(x')\right)\right\}\left(\frac{1}{n}\sum_{i=1}^n\xi_{it}^2-1\right)\phi_t(x)\phi_t(x')|\mathcal{F}_{t-1}\right]\\
=\;& 2(E\xi_{1t}^3-1)E\left[\left\{\sum_{k=1}^{t-1}\bar{\xi}_{\cdot k}\left(\phi_k(x)\phi_t(x')+\phi_t(x)\phi_k(x')\right)\right\}\phi_t(x)\phi_t(x')|\mathcal{F}_{t-1}\right],
\end{aligned}$$

where $\bar{\xi}_{\cdot k} = \frac{1}{n}\sum_{i=1}^{n}\xi_{ik}$. Note that

$$\sup_{(x,x')\in[0,1]^2}\sum_{k=1}^{\infty}\sum_{k'=1}^{\infty}\{|\phi_{k'}(x)\phi_k(x')\phi_k(x)\phi_k(x')| + |\phi_k(x)\phi_{k'}(x')\phi_k(x)\phi_k(x')|\} < \infty,$$

thus,

$$\sum_{t=2}^{k_n} v_t^{(3)}(x,x') = 2(E\xi_{1t}^3 - 1)\sum_{t=2}^{k_n}\sum_{k=1}^{t-1} E\left[\bar{\xi}_{\cdot k}|\mathcal{F}_{t-1}\right]\{\phi_k(x)\phi_t(x') + \phi_t(x)\phi_k(x')\}\phi_t(x)\phi_t(x')$$

converges to zero in probability as $n\to\infty$ and $k_n\to\infty$.

According to (S.1), one has, as $n\to\infty$ and $k_n\to\infty$,

$$\begin{aligned}\sum_{t=2}^{k_n} E\{d_t^2(x,x')|\mathcal{F}_{t-1}\} &\to \sum_{k\neq k'}^{\infty}\phi_k^2(x)\phi_{k'}^2(x') + \sum_{k\neq k'}^{\infty}\phi_k(x)\phi_{k'}(x')\phi_k(x')\phi_{k'}(x) \\ &\quad + \sum_{k=1}^{\infty}\left(E\xi_{1k}^4 - 1\right)\phi_k^2(x)\phi_k^2(x') \\ &= G^2(x,x') + G(x,x)G(x',x') + \sum_{k=1}^{\infty}\left(E\xi_{1k}^4 - 3\right)\phi_k^2(x)\phi_k^2(x'),\end{aligned}$$

in probability.

In the following, denote

$$E\left\{d_t^3(x,x')|\mathcal{F}_{t-1}\right\} = w_t^{(1)}(x,x') + 3w_t^{(2)}(x,x') + 3w_t^{(3)}(x,x') + w_t^{(4)}(x,x')$$

where $w_t^{(i)}(x,x')$, $i=1,2,3,4$ are defined in the following:

$$\begin{aligned}w_t^{(1)}(x,x') &= n^{3/2}E\left[\left\{\sum_{k=1}^{t-1}\bar{\xi}_{\cdot k,t}\left(\phi_k(x)\phi_t(x') + \phi_t(x)\phi_k(x')\right)\right\}^3\middle|\mathcal{F}_{t-1}\right], \\ w_t^{(2)}(x,x') &= n^{3/2}E\left[\left\{\sum_{k=1}^{t-1}\bar{\xi}_{\cdot k,t}\left(\phi_k(x)\phi_t(x') + \phi_t(x)\phi_k(x')\right)\right\}^2\left(\bar{\xi}_{\cdot tt} - 1\right)\phi_t(x)\phi_t(x')\middle|\mathcal{F}_{t-1}\right], \\ w_t^{(3)}(x,x') &= n^{3/2}E\left[\left\{\sum_{k=1}^{t-1}\bar{\xi}_{\cdot k,t}\left(\phi_k(x)\phi_t(x') + \phi_t(x)\phi_k(x')\right)\right\}\left(\bar{\xi}_{\cdot tt} - 1\right)^2\phi_t^2(x)\phi_t^2(x')\middle|\mathcal{F}_{t-1}\right], \\ w_t^{(4)}(x,x') &= n^{3/2}E\left[\left\{\left(\bar{\xi}_{\cdot tt} - 1\right)\phi_t(x)\phi_t(x')\right\}^3|\mathcal{F}_{t-1}\right].\end{aligned}$$

Then one has for any $(x, x') \in [0,1]^2$,

$$\begin{aligned}
& w_t^{(1)}(x, x') \\
= \;& n^{3/2} E\left[\left\{\sum_{1\le k\le t-1} \bar{\xi}_{\cdot kt}\phi_k(x)\phi_t(x') + \sum_{1\le k'\le t-1} \bar{\xi}_{\cdot k't}\phi_t(x)\phi_{k'}(x')\right\}^3 \middle| \mathcal{F}_{t-1}\right] \\
= \;& n^{-3/2} E\left[\left\{\sum_{1\le k\le t-1}\sum_{i=1}^{n} \xi_{ik}\xi_{it}\phi_k(x)\phi_t(x')\right.\right. \\
& \left.\left. + \sum_{1\le k'\le t-1}\sum_{i=1}^{n} \xi_{ik'}\xi_{it}\phi_{k'}(x')\phi_t(x)\right\}^3 \middle| \mathcal{F}_{t-1}\right] \\
= \;& n^{-3/2} E\left[\sum_{i=1}^{n}\left\{\left(\sum_{1\le k\le t-1} \xi_{ik}^3\phi_k^3(x) \; + 3\sum_{1\le k,k'\le t-1} \xi_{ik}^2\phi_k^2(x)\xi_{ik'}\phi_{k'}(x)\right)\xi_{it}^3\phi_t^3(x')\right.\right. \\
& \left.\left. + \left(\sum_{1\le k'\le t-1} \xi_{ik'}^3\phi_{k'}^3(x') \; +3\sum_{1\le k,k'\le t-1} \xi_{ik}^2\phi_k^2(x')\xi_{ik'}\phi_{k'}(x')\right)\xi_{it}^3\phi_t^3(x)\right\} \middle| \mathcal{F}_{t-1}\right] \\
= \;& n^{-1/2}\left(E\xi_{1t}^3\right)\left[E\left\{\sum_{1\le k\le t-1}\left(\frac{1}{n}\sum_{i=1}^{n}\xi_{ik}^3\right)\left(\phi_k^3(x)\phi_t^3(x') + \phi_k^3(x')\phi_t^3(x)\right) \middle| \mathcal{F}_{t-1}\right\}\right. \\
& \left. +3E\left\{\sum_{1\le k,k'\le t-1}\left(\frac{1}{n}\sum_{i=1}^{n}\xi_{ik}^2\xi_{ik'}\right)\left(\phi_k^2(x)\phi_{k'}(x)\phi_t^3(x') + \phi_k^2(x')\phi_{k'}(x')\phi_t^3(x)\right) \middle| \mathcal{F}_{t-1}\right\}\right]
\end{aligned}$$

Next note that

$$\sup_{(x,x')\in[0,1]^2}\sum_{k=1}^{\infty}\sum_{k'=1}^{\infty}\{|\phi_{k'}^3(x)\phi_k^3(x')| + |\phi_k^3(x)\phi_{k'}^3(x')|\} < \infty,$$

$$\sup_{(x,x')\in[0,1]^2}\sum_{k=1}^{\infty}\sum_{k'=1}^{\infty}\sum_{k''=1}^{\infty}\{|\phi_k^2(x)\phi_{k'}(x)\phi_{k''}^3(x')| + |\phi_k^2(x')\phi_{k'}(x')\phi_{k''}^3(x)|\} < \infty.$$

Hence, as $n \to \infty$ and $k_n \to \infty$,

$$\begin{aligned}
& \sum_{t=2}^{k_n} w_t^{(1)}(x, x') \\
= \;& n^{-1/2}\left(E\xi_{1t}^3\right)\left[\sum_{t=2}^{k_n} E\left\{\sum_{k=1}^{t-1}\left(\frac{1}{n}\sum_{i=1}^{n}\xi_{ik}^3\right)\left(\phi_k^3(x)\phi_t^3(x') + \phi_k^3(x')\phi_t^3(x)\right) \middle| \mathcal{F}_{t-1}\right\}\right. \\
& \left. +3\sum_{t=2}^{k_n} E\left\{\sum_{1\le k,k'\le t-1}\left(\frac{1}{n}\sum_{i=1}^{n}\xi_{ik}^2\xi_{ik'}\right)\left(\phi_k^2(x)\phi_{k'}(x)\phi_t^3(x') + \phi_k^2(x')\phi_{k'}(x')\phi_t^3(x)\right) \middle| \mathcal{F}_{t-1}\right\}\right] \\
\to \;& 0, \text{ in probability.}
\end{aligned}$$

For $w_t^{(2)}(x,x')$, one has

$$\begin{aligned}
& w_t^{(2)}(x,x') \\
= \;& n^{1/2} E\left[\frac{1}{n}\sum_{i=1}^n \left\{\sum_{1\le k\le t-1} \xi_{ik}\left(\phi_k(x)\phi_t(x')+\phi_t(x)\phi_k(x')\right)\right\}^2\right. \\
& \left.\xi_{it}^2\left(\bar{\xi}_{\cdot tt}-1\right)\phi_t(x)\phi_t(x')\right|\mathcal{F}_{t-1}\Big\} \\
= \;& n^{1/2} E\left\{\sum_{1\le k\le t-1}\frac{1}{n}\sum_{i=1}^n \xi_{it}^2\left(\frac{1}{n}\sum_{i=1}^n \xi_{it}^2-1\right)\xi_{ik}^2\left(\phi_k(x)\phi_t(x')\right.\right. \\
& \left.\left.+\phi_t(x)\phi_k(x')\right)^2\phi_t(x)\phi_t(x')\right|\mathcal{F}_{t-1}\Big\} \\
= \;& n^{1/2} E\left\{\sum_{1\le k\le t-1}\frac{1}{n}\left[\sum_{i=1}^n\left(\frac{1}{n}\xi_{it}^4-\xi_{it}^2\right)\xi_{ik}^2+\frac{1}{n}\sum_{i\ne j}\xi_{it}^2\xi_{jt}^2\xi_{ik}^2\right]\left(\phi_k(x)\phi_t(x')\right.\right. \\
& \left.\left.+\phi_t(x)\phi_k(x')\right)^2\phi_t(x)\phi_t(x')\right|\mathcal{F}_{t-1}\Big\} \\
= \;& n^{-1/2}(E\xi_{1t}^4-1)\sum_{1\le k\le t-1} E\left\{\bar{\xi}_{\cdot kk}\left(\phi_k^2(x)\phi_t^3(x')\phi_t(x)+\phi_k^2(x')\phi_t^3(x)\phi_t(x')\right.\right. \\
& \left.\left.+2\phi_k(x)\phi_k(x')\phi_t^2(x')\phi_t^2(x)\right)\right|\mathcal{F}_{t-1}\Big\}.
\end{aligned}$$

In fact,

$$\begin{aligned}
\sum_{t=2}^{k_n} w_t^{(2)}(x,x') \;=\;& n^{-1/2}(E\xi_{1t}^4-1)\sum_{t=2}^{k_n}\sum_{k=1}^{t-1} E\left\{\bar{\xi}_{\cdot kk}\left(\phi_k^2(x)\phi_t^3(x')\phi_t(x)+\phi_k^2(x')\phi_t^3(x)\phi_t(x')\right.\right. \\
& \left.\left.+2\phi_k(x)\phi_k(x')\phi_t^2(x')\phi_t^2(x)\right)\right|\mathcal{F}_{t-1}\Big\},
\end{aligned}$$

and

$$\sup_{(x,x')\in[0,1]^2}\sum_{k=1}^\infty\sum_{k'=1}^\infty\{|\phi_{k'}^2(x)\phi_k^3(x')\phi_k(x)|+|\phi_{k'}^2(x')\phi_k^3(x')\phi_k(x')|+|\phi_{k'}(x)\phi_{k'}(x')\phi_k^2(x')\phi_k^2(x)|\}<\infty,$$

Therefore, for any $(x,x')\in[0,1]^2$,

$$\sum_{t=2}^{k_n} w_t^{(2)}(x,x')\to 0 \text{ in probability.}$$

Note that

$$\begin{aligned}
& w_t^{(3)}(x,x') \\
= \;& n^{3/2}E\left\{\frac{1}{n}\sum_{j=1}^n\left[\frac{1}{n^2}\left(\sum_{i=1}^n\xi_{it}^2\right)^2+1-\frac{2}{n}\sum_{i=1}^n\xi_{it}^2\right]\xi_{jt}\xi_{jk}\left(\phi_k(x)\phi_t(x')+\phi_t(x)\phi_k(x')\right)\right. \\
& \left.\times\;\phi_t^2(x)\phi_t^2(x')\right|\mathcal{F}_{t-1}\Big\} \\
= \;& n^{-1/2}\left[E(\xi_{1t}^5)-2E(\xi_{1t}^3)\right]\sum_{1\le k\le t-1} E\left\{\bar{\xi}_{\cdot k}\left(\phi_k(x)\phi_t(x')+\phi_t(x)\phi_k(x')\right)\phi_t^2(x)\phi_t^2(x')\right|\mathcal{F}_{t-1}\Big\}
\end{aligned}$$

Hence, for any $(x,x') \in [0,1]^2$

$$\sum_{t=2}^{k_n} w_t^{(3)}(x,x') = n^{-1/2}\left[E(\xi_{1t}^5) - 2E(\xi_{1t}^3)\right]$$
$$\times \sum_{t=2}^{k_n}\sum_{k=1}^{t-1} E\left\{ \frac{1}{n}\sum_{i=1}^{n} \xi_{ik}\left(\phi_k(x)\phi_t(x') + \phi_t(x)\phi_k(x')\right)\phi_t^2(x)\phi_t^2(x') \middle| \mathcal{F}_{t-1}\right\}$$

and $\sum_{t=2}^{k_n} w_t^{(3)}(x,x') \to 0$ in probability.

Next for any $(x,x') \in [0,1]^2$, under assumptions (A4) and (A5), one has

$$\begin{aligned} w_t^{(4)}(x,x') &= E\left[\left\{\sqrt{n}\left(\bar{\xi}_{\cdot tt} - 1\right)\phi_t(x)\phi_t(x')\right\}^3 |\mathcal{F}_{t-1}\right] \\ &= \frac{1}{\sqrt{n}}\phi_t^3(x)\phi_t^3(x')E\left(\xi_{1t}^2 - 1\right)^3. \end{aligned}$$

Therefore, as $n \to \infty$ and $k_n \to \infty$, $\sup_{(x,x')\in[0,1]^2} \sum_{t=2}^{k_n} w_t^{(4)}(x,x') \to 0$, in probability.

According to the definition of $E\left[d_t^3(x,x')|\mathcal{F}_{t-1}\right]$, one has

$$\sup_{(x,x')\in[0,1]^2} \sum_{t=2}^{k_n} E\left[d_t^3(x,x') I(\left|d_t^2(x,x')\right| > \epsilon)|\mathcal{F}_{t-1}\right] \to 0, \text{ in probability, for every } \epsilon > 0.$$

The uniform central limit theorem, Theorem 1 in Bae & Choi (1999) induces that $\mathcal{L}(S_t(x,x'),(x,x') \in [0,1]^2)$ converges to $\mathcal{L}(\zeta_Z(x,x'), (x,x') \in [0,1]^2)$, as $n \to \infty$, where $\zeta_Z(x,x')$ is a gaussian process such that $E\zeta_Z(x,x') = 0$,

$$E\zeta_Z^2(x,x') = G^2(x,x') + G(x,x)G(x',x') + \sum_{k=1}^{\infty}\left(E\xi_{1k}^4 - 3\right)\phi_k^2(x)\phi_k^2(x')$$

and covariance function

$$\begin{aligned} \operatorname{cov}\left(\zeta_Z(x,x'), \zeta_Z(y,y')\right) &= \sum_{k\neq k'}^{\infty} \phi_k^2(x)\,\phi_{k'}^2(x')\,\phi_k^2(y)\,\phi_{k'}^2(y') \\ &+ \sum_{k,k'=1}^{\infty}\left(E\xi_{1k}^4 - 1\right)\phi_k^2(x)\,\phi_{k'}^2(x')\,\phi_k^2(y)\,\phi_{k'}^2(y'). \end{aligned}$$

According to the definition of $\zeta(x,x')$ in Section 3.2, Lemma 6 is proved.

S2 Proof of Proposition 1

Recall that the error terms defined in Section 2 are $U_{ij} = Y_{ij} - m(j/N)$, $i = 1, ..., n$, $j = 1, ..., N$. Note that

$$\begin{aligned}
\bar{U}_{\cdot jj'} &= n^{-1} \sum_{i=1}^{n} U_{ij} U_{ij'} \\
&= n^{-1} \sum_{i=1}^{n} \left\{ \sum_{k=1}^{\infty} \xi_{ik} \phi_k (j/N) + \sigma (j/N) \varepsilon_{ij} \right\} \left\{ \sum_{k=1}^{\infty} \xi_{ik} \phi_k (j'/N) + \sigma (j'/N) \varepsilon_{ij'} \right\} \\
&= \bar{U}_{1jj'} + \bar{U}_{2jj'} + \bar{U}_{3jj'} + \bar{U}_{4jj'},
\end{aligned}$$

where

$$\begin{aligned}
\bar{U}_{1jj'} &= \sum_{k \neq k'}^{\infty} \bar{\xi}_{\cdot kk'} \phi_k (j/N) \phi_{k'} (j'/N), \\
\bar{U}_{2jj'} &= \sum_{k=1}^{\infty} \bar{\xi}_{\cdot kk} \phi_k (j/N) \phi_k (j'/N), \\
\bar{U}_{3jj'} &= n^{-1} \sum_{i=1}^{n} \sigma (j/N) \sigma (j'/N) \varepsilon_{ij} \varepsilon_{ij'}, \\
\bar{U}_{4jj'} &= n^{-1} \sum_{i=1}^{n} \left\{ \sum_{k=1}^{\infty} \xi_{ik} \phi_k (j/N) \sigma (j'/N) \varepsilon_{ij'} + \sum_{k=1}^{\infty} \xi_{ik} \phi_k (j'/N) \sigma (j/N) \varepsilon_{ij} \right\}.
\end{aligned}$$

Let $\tilde{\mathcal{U}}_{ip_2}(x, x') = \boldsymbol{B}_{p_2}^{\mathrm{T}}(x, x') \left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1} \mathbf{X}^{\mathrm{T}} \bar{\mathbf{U}}_i$, where $\bar{\mathbf{U}}_i = \left\{\bar{U}_{ijj'}\right\}_{1 \leq j \neq j' \leq N}$, for $i = 1, 2, 3, 4$. Then we can have the following decomposition for $\tilde{G}_{p_2}(x, x')$

$$\tilde{G}_{p_2}(x, x') = \tilde{\mathcal{U}}_{1p_2}(x, x') + \tilde{\mathcal{U}}_{2p_2}(x, x') + \tilde{\mathcal{U}}_{3p_2}(x, x') + \tilde{\mathcal{U}}_{4p_2}(x, x').$$

Define

$$\mathcal{U}_1(x, x') = \sum_{k \neq k'}^{\infty} \bar{\xi}_{\cdot kk'} \phi_k (x) \phi_{k'} (x'),$$

$$\mathcal{U}_2(x, x') = G(x, x') + \sum_{k=1}^{\infty} \left\{ \phi_k (x) \phi_k (x') \left(\bar{\xi}_{\cdot kk} - 1\right) \right\}.$$

Next we illustrate the facts that $\tilde{\mathcal{U}}_{1p_2}(x, x')$ and $\tilde{\mathcal{U}}_{2p_2}(x, x')$ are the dominating terms in the above decomposition, which converge uniformly to $\mathcal{U}_1(x, x')$ and $\mathcal{U}_2(x, x')$ respectively, while $\tilde{\mathcal{U}}_{3p_2}(x, x')$ and $\tilde{\mathcal{U}}_{4p_2}(x, x')$ are negligible noise terms.

By the definition of $\bar{\mathbf{U}}_1$, one has that

$$\begin{aligned}
\tilde{\mathcal{U}}_{1p_2}(x, x') &= N^{-2} \boldsymbol{B}_{p_2}^{\mathrm{T}}(x, x') \hat{\mathbf{V}}_{p_2,2}^{-1} \mathbf{X}^{\mathrm{T}} \left\{ \sum_{k \neq k'}^{\infty} \bar{\xi}_{\cdot kk'} \phi_k (j/N) \phi_{k'} (j'/N) \right\} \\
&= \sum_{k \neq k'}^{\infty} \bar{\xi}_{\cdot kk'} \widetilde{\boldsymbol{\phi}}_{kk'} (x, x').
\end{aligned}$$

For any $1 \leq k \neq k' \leq \infty$, $E\left(\bar{\xi}_{\cdot kk'}\right)^2 \leq n^{-1}\left(E\xi_{1k}^4 E\xi_{1k'}^4\right)^{1/2} = O\left(n^{-1}\right)$. Lemma 5 and Assumption (A3) imply that

$$
\begin{aligned}
&E\left\{\sup_{x,x'\in[0,1]^2}\left|\sum_{k\neq k'}^{\infty}\bar{\xi}_{\cdot kk'}(\boldsymbol{\phi}_{kk'}-\widetilde{\boldsymbol{\phi}}_{kk'})(x,x')\right|\right\}^2 \\
&\quad\leq \sum_{k\neq k'}^{\infty} E\bar{\xi}_{\cdot kk'}^2\left\{\sup_{x,x'\in[0,1]^2}\left|(\boldsymbol{\phi}_{kk'}-\widetilde{\boldsymbol{\phi}}_{kk'})(x,x')\right|\right\}^2 \\
&\quad\leq Cn^{-1}\sum_{k=1}^{\infty}\sum_{k'=1}^{\infty} ch_{s_2}^{2p_2}\left(\|\phi_{k'}\|_\infty\|\phi_k\|_{0,1}\right)^2 \\
&\quad\leq Cn^{-1}h_{s_2}^{2p_2}\sum_{k=1}^{\infty}\|\phi_k\|_{0,1} = o\left(n^{-1}\right),
\end{aligned}
$$

thus,

$$
\begin{aligned}
\sup_{x,x'\in[0,1]^2}\left|(\tilde{\mathcal{U}}_{1p_2}-\mathcal{U}_1)(x,x')\right| &= \sup_{x,x'\in[0,1]^2}\left|\left\{\tilde{\mathcal{U}}_{1p_2}-\sum_{k\neq k'}^{\infty}\bar{\xi}_{\cdot kk'}\widetilde{\boldsymbol{\phi}}_{kk'}\right\}(x,x')\right| \\
= \sup_{x,x'\in[0,1]^2}&\left|\sum_{k\neq k'}^{\infty}\bar{\xi}_{\cdot kk'}(\boldsymbol{\phi}_{kk'}-\widetilde{\boldsymbol{\phi}}_{kk'})(x,x')\right| \\
= o_p(n^{-1/2}).&
\end{aligned}
$$

Similarly, one has $E\left(\bar{\xi}_{\cdot kk}-1\right)^2 = n^{-1}\left(E\xi_{1k}^4-1\right)$. Also

$$
\begin{aligned}
\sup_{x,x'\in[0,1]^2}\left|(\tilde{\mathcal{U}}_{2p_2}-\mathcal{U}_2)(x,x')\right| &= \sup_{x,x'\in[0,1]^2}\left|\sum_{k=1}^{\infty}\left(\bar{\xi}_{\cdot kk}-1\right)(\boldsymbol{\phi}_{kk}-\widetilde{\boldsymbol{\phi}}_{kk})(x,x')\right| \\
&= o_p\left(n^{-1/2}\right).
\end{aligned}
$$

Lemma 5 and Assumption (A3) imply that

$$
\begin{aligned}
&E\left\{\sup_{x,x'\in[0,1]^2}\left|\sum_{k=1}^{\infty}\left(\bar{\xi}_{\cdot kk}-1\right)(\boldsymbol{\phi}_{kk}-\widetilde{\boldsymbol{\phi}}_{kk})(x,x')\right|\right\}^2 \\
&\quad\leq \sum_{k=1}^{\infty}E\left(\bar{\xi}_{\cdot kk}-1\right)^2\left\{\sup_{x,x'\in[0,1]^2}\left|(\boldsymbol{\phi}_{kk'}-\widetilde{\boldsymbol{\phi}}_{kk'})(x,x')\right|\right\}^2 \\
&\quad\leq Cn^{-1}h_{s_2}^{2p_2}\sum_{k=1}^{\infty}\|\phi_k\|_{0,1} = o\left(n^{-1}\right),
\end{aligned}
$$

then $\sup_{x,x'\in[0,1]^2}\left|(\tilde{\mathcal{U}}_{2p_2}-\mathcal{U}_2)(x,x')\right| = o_p\left(n^{-1/2}\right)$. Therefore, one has

$$
\sup_{x,x'\in[0,1]^2}\left|(\tilde{\mathcal{U}}_{1p_2}+\tilde{\mathcal{U}}_{2p_2}-\mathcal{U}_1-\mathcal{U}_2)(x,x')\right| = o_p\left(n^{-1/2}\right).
$$

Denote that

$$N^{-2}\mathbf{X}^{\mathrm{T}}\bar{\mathbf{U}}_3 = \left\{n^{-1}\sum_{i=1}^{n} A_{iJJ'}\right\}_{J,J'=1-p_2}^{N_{s_2}},$$

where

$$A_{iJJ'} = N^{-2}\sum_{1\le j\ne j'\le N} B_{J,p_2}\left(j/N\right)\sigma\left(j/N\right)B_{J',p_2}\left(j'/N\right)\sigma\left(j'/N\right)\varepsilon_{ij}\varepsilon_{ij'}.$$

It is easy to see that $EA_{iJJ'}=0$ and $EA_{iJJ'}^2 = O\left(h_{s_2}^2 N^{-2}\right)$. Using standard arguments in Wang and Yang (2009), one has

$$\left\|N^{-2}\mathbf{X}^{\mathrm{T}}\bar{\mathbf{U}}_3\right\|_\infty = o_{\text{a.s.}}\left\{N^{-1}n^{-1/2}h_{s_2}\log^{1/2}(n)\right\}.$$

Therefore, according to the definition of $\tilde{\tilde{U}}_{3p_2}(x,x')$, one has

$$\begin{aligned}
&\sup_{x,x'\in[0,1]^2}\left\|\boldsymbol{B}_{p_2}^{\mathrm{T}}(x,x')\hat{\mathbf{V}}_{p_2,2}^{-1}N^{-2}\mathbf{X}^{\mathrm{T}}\bar{\mathbf{U}}_3\right\|_\infty \\
&\qquad\le C_{p_2}\sup_{x,x'\in[0,1]^2}\left\|\mathbf{B}_{p_2}(x,x')\right\|_\infty\left\|\hat{\mathbf{V}}_{p_2,2}^{-1}N^{-2}\mathbf{X}^{\mathrm{T}}\bar{\mathbf{U}}_3\right\|_\infty \\
&\qquad= o_{\text{a.s.}}\left\{N^{-1}n^{-1/2}h_{s_2}^{-1}\log^{1/2}n\right\} = o_{\text{a.s.}}\left(n^{-1/2}\right).
\end{aligned}$$

Likewise, in order to get the upper bound of $\left|\tilde{\mathcal{U}}_{4p_2}(x,x')\right|$, one has

$$\mathbf{X}^{\mathrm{T}}\bar{\mathbf{U}}_4 = \frac{2}{n}\left\{\sum_{i=1}^{n}\sum_{k=1}^{\infty}\xi_{ik}\sum_{1\le j\ne j'\le N}\phi_k\left(j/N\right)\sigma\left(j'/N\right)B_{JJ',p_2}\left(j/N,j'/N\right)\varepsilon_{ij'}\right\}_{J,J'=1-p_2}^{N_{s_2}}.$$

Let

$$D_{iJJ'} = N^{-2}\sum_{k=1}^{\infty}\left\{\xi_{ik}\sum_{1\le j\ne j'\le N}\phi_k\left(j/N\right)\sigma\left(j'/N\right)B_{JJ',p}\left(j/N,j'/N\right)\varepsilon_{ij'}\right\},$$

then $ED_{iJJ'}=0$,

$$\begin{aligned}
ED_{iJJ'}^2 &= N^{-4}\sum_{k=1}^{\infty}\sum_{1\le j\ne j'\le N}\phi_k^2\left(j/N\right)\sigma^2\left(j'/N\right)B_{JJ',p_2}^2\left(j/N,j'/N\right)E\xi_{ik}^2E\varepsilon_{ij'}^2 \\
&\le CN^{-4}\left\{\sup_{x\in[0,1]}\sum_{k=1}^{\infty}\phi_k^2(x)\right\}\sum_{1\le j\ne j'\le N}B_{J,p_2}^2\left(j/N\right)\sigma^2\left(j'/N\right)B_{J',p_2}^2\left(j'/N\right) \\
&= CN^{-4}\left\{\sup_{x\in[0,1]}G(x,x)\right\}\sum_{1\le j\ne j'\le N}B_{J,p_2}^2\left(j/N\right)\sigma^2\left(j'/N\right)B_{J',p_2}^2\left(j'/N\right) \\
&= O\left(h_{s_2}^2N^{-2}\right).
\end{aligned}$$

Similar arguments in Wang and Yang (2009) leads to

$$\left\| \frac{\mathbf{X}^{\mathrm{T}}\bar{\mathbf{U}}_4}{N^2} \right\|_\infty = o_{\mathrm{a.s.}} \left\{ N^{-1} n^{-1/2} h_{\mathrm{s}_2} \log^{1/2}(n) \right\}.$$

Thus,

$$\begin{aligned} \left\| \boldsymbol{B}_{p_2}^{\mathrm{T}}(x,x') \hat{\mathbf{V}}_{p_2,2}^{-1} \frac{\mathbf{X}^{\mathrm{T}}\bar{\mathbf{U}}_4}{N^2} \right\|_\infty &= o_{\mathrm{a.s.}} \left\{ N^{-1} n^{-1/2} h_{\mathrm{s}_2}^{-1} \log^{1/2}(n) \right\} \\ &= o_{\mathrm{a.s.}} \left(n^{-1/2} \right). \end{aligned}$$

S3 Proof of Proposition 2

For simplicity, denote

$$\mathbf{X}_1 = \begin{pmatrix} B_{1-p_1,p_1}(1/N) & \cdots & B_{N_{\mathrm{s}_1},p_1}(1/N) \\ \cdots & \cdots & \cdots \\ B_{1-p_1,p_1}(N/N) & \cdots & B_{N_{\mathrm{s}_1},p_1}(N/N) \end{pmatrix}_{N \times \left(N_{\mathrm{s}_1}+p_1\right)}$$

for the positive integer p_1. We decompose $\hat{m}_{p_1}(j/N)$ into three terms $\tilde{m}_{p_1}(j/N)$, $\tilde{\xi}_{p_1}(j/N)$ and $\tilde{\varepsilon}_{p_1}(j/N)$ in the space $\mathcal{H}^{(p_1-2)}$ of spline functions:

$$\hat{m}_{p_1}(x) = \tilde{m}_{p_1}(x) + \tilde{\varepsilon}_{p_1}(x) + \tilde{\xi}_{p_1}(x),$$

where

$$\begin{aligned} \tilde{m}_p(x) &= \{B_{1-p_1,p_1}(x), \ldots, B_{N_{\mathrm{s}},p_1}(x)\} \left(\mathbf{X}_1^{\mathrm{T}}\mathbf{X}_1\right)^{-1} \mathbf{X}_1^{\mathrm{T}}\mathbf{m}, \\ \tilde{\varepsilon}_{p_1}(x) &= \{B_{1-p_1,p_1}(x), \ldots, B_{N_{\mathrm{s}},p_1}(x)\} \left(\mathbf{X}_1^{\mathrm{T}}\mathbf{X}_1\right)^{-1} \mathbf{X}_1^{\mathrm{T}}\mathbf{e}, \\ \tilde{\xi}_{p_1}(x) &= \{B_{1-p_1,p_1}(x), \ldots, B_{N_{\mathrm{s}},p_1}(x)\} \left(\mathbf{X}_1^{\mathrm{T}}\mathbf{X}_1\right)^{-1} \mathbf{X}_1^{\mathrm{T}} \sum_{k=1}^{\infty} \bar{\xi}_{\cdot k} \boldsymbol{\phi}_k, \end{aligned}$$

where $\mathbf{m} = (m(1/N), \ldots, m(N/N))^{\mathrm{T}}$ is the signal vector, $\mathbf{e} = (\sigma(1/N)\bar{\varepsilon}_{\cdot 1}, \ldots, \sigma(N/N)\bar{\varepsilon}_{\cdot N})^{\mathrm{T}}$, $\bar{\varepsilon}_{\cdot j} = n^{-1}\sum_{i=1}^{n} \varepsilon_{ij}$, $1 \le j \le N$, is the noise vector and $\boldsymbol{\phi}_k = (\phi_k(1/N), \ldots, \phi_k(N/N))^{\mathrm{T}}$ are the eigenfunction vectors, and $\bar{\xi}_{\cdot k} = n^{-1}\sum_{i=1}^{n} \xi_{ik}$, $1 \le k \le \infty$.

Thus, one can write the residuals $\hat{U}_{ij,p_1} = Y_{ij} - \hat{m}_{p_1}(j/N)$ as

$$\hat{U}_{ij,p_1} = m(j/N) - \tilde{m}_{p_1}(j/N) - \tilde{\xi}_{p_1}(j/N) - \tilde{\varepsilon}_{p_1}(j/N) + U_{ij}.$$

Let $\hat{\bar{U}}_{\cdot jj',p_1} = n^{-1}\sum_{i=1}^{n} \hat{U}_{ij,p_1}\hat{U}_{ij,'p_1}$. We calculate the difference of $\hat{G}_{p_1,p_2}(x,x') - \tilde{G}_{p_2}(x,x')$ by checking the difference $\hat{\bar{U}}_{\cdot jj',p_1} - \bar{U}_{\cdot jj'}$ first. For any $1 \le j \ne j' \le N$, one has

$$\begin{aligned} \hat{\bar{U}}_{\cdot jj',p_1} - \bar{U}_{\cdot jj'} &= n^{-1}\sum_{i=1}^{n} [U_{ij}(\tilde{m}_{p_1} - m)(j'/N) + U_{ij'}(\tilde{m}_{p_1} - m)(j/N) \\ &\quad + U_{ij}\tilde{\xi}_{p_1}(j'/N) + U_{ij'}\tilde{\xi}_{p_1}(j/N) + U_{ij}\tilde{\varepsilon}_{p_1}(j'/N) + U_{ij'}\tilde{\varepsilon}_{p_1}(j/N)] \\ &\quad + (\hat{m}_{p_1} - m)(j'/N)(\hat{m}_{p_1} - m)(j/N). \end{aligned}$$

Next, we calculate the super norm of each part of $\hat{G}_{p_1,p_2}(x,x') - \tilde{G}_{p_2}(x,x')$ respectively. One can write $\tilde{\varepsilon}_{p_1}(j'/N) = \sum_{J=1-p_1}^{N_{s_1}} B_{J,p_1}(j'/N)\, w_J$, where $\{w_J\}_{J=1-p_1}^{N_{s_1}} = N^{-1}\hat{\mathbf{V}}_{p_1}^{-1}\mathbf{X}_1^{\mathrm{T}}\mathbf{e}$ and

$$\begin{aligned} &\mathbf{X}^{\mathrm{T}}\left\{\sum_{i=1}^{n} U_{ij}\tilde{\varepsilon}_{p_1}(j'/N)\right\}_{1\le j\ne j'\le N} \\ = &\left\{\sum_{i=1}^{n}\sum_{1\le j\ne j'\le N} U_{ij}\tilde{\varepsilon}_{p_1}(j'/N)\, B_{JJ',p_2}(j/N,j'/N)\right\}_{J,J'=1-p_2}^{N_{s_2}}. \end{aligned}$$

Thus,

$$\begin{aligned} &\left\|\mathbf{X}^{\mathrm{T}}\left\{n^{-1}\sum_{i=1}^{n} U_{ij}\tilde{\varepsilon}_{p_1}(j'/N)\right\}_{1\le j\ne j'\le N}\right\|_2^2 \\ &= \sum_{J,J'=1-p_2}^{N_{s_2}}\left[\frac{1}{n}\sum_{i=1}^{n}\sum_{1\le j\ne j'\le N} U_{ij}\left(\sum_{J''=1-p_1}^{N_{s_1}} B_{J'',p_1}(j'/N)\, w_{J''}\right) B_{JJ',p_2}(j/N,j'/N)\right]^2 \\ &\le \sum_{J,J'=1-p_2}^{N_{s_2}}\sum_{J''=1-p_1}^{N_{s_1}}\left[\frac{1}{n}\sum_{i=1}^{n}\sum_{1\le j\ne j'\le N} U_{ij}B_{J'',p_1}\left(\frac{j'}{N}\right)B_{JJ',p_2}\left(\frac{j}{N},\frac{j'}{N}\right)\right]^2\sum_{J''=1-p_1}^{N_{s_1}} w_{J''}^2 \\ &= I\times II, \end{aligned}$$

where

$$I = \sum_{J,J'=1-p_2}^{N_{s_2}}\sum_{J''=1-p_1}^{N_{s_1}}\left\{n^{-1}\sum_{i=1}^{n}\sum_{1\le j\ne j'\le N} U_{ij}B_{JJ',p_2}(j/N,j'/N)\, B_{J'',p_1}(j'/N)\right\}^2$$

and $II = \left\|N^{-1}\hat{\mathbf{V}}_{p_1}^{-1}\mathbf{X}_1^{\mathrm{T}}\mathbf{e}\right\|_2^2$.

Let $h_* = \min\{h_{s_1}, h_{s_2}\}$. The definition of spline function implies that

$$\begin{aligned} E[I] &= n^{-1}\sum_{J,J'=1-p_2}^{N_{s_2}}\sum_{J''=1-p_1}^{N_{s_1}}\left\{\sum_{j,j''=1}^{N} E(U_{1j}U_{1j''})\, B_{J,p_2}(j/N)\, B_{J,p_2}(j''/N)\right. \\ &\qquad \left.\times\sum_{j'\ne j}^{N}\sum_{j'''\ne j''}^{N} B_{J',p_2}(j'/N)\, B_{J'',p_1}(j'/N)\, B_{J',p_2}(j'''/N)\, B_{J'',p_1}(j'''/N)\right\} \\ &\le C(G,\sigma^2)n^{-1}N^4h_{s_2}^2h_*^2N_{s_2}\max\{N_{s_1},N_{s_2}\} \le C(G,\sigma^2)n^{-1}N^4h_{s_2}h_*. \end{aligned}$$

Hence,

$$\left\|N^{-2}\mathbf{X}^{\mathrm{T}}\left\{n^{-1}\sum_{i=1}^{n} U_{ij}\tilde{\varepsilon}_{p_1}(j'/N)\right\}_{1\le j\ne j'\le N}\right\|_2 = O\left(n^{-1/2}h_{s_2}^{1/2}h_*^{1/2}\right).$$

Meanwhile, one has that

$$II \leq C_{p_1} \left\| N^{-1}\mathbf{X}_1^{\mathrm{T}}\mathbf{e} \right\|_2^2 h_{\mathrm{s}_1}^{-2} = O_{\mathrm{a.s.}}\left\{ (Nnh_{\mathrm{s}_1}^2)^{-1} \right\}.$$

By Lemma 4, one has

$$\begin{aligned} & n^{1/2} \sup_{x,x' \in [0,1]^2} \left\| \boldsymbol{B}_{p_2}^{\mathrm{T}}(x,x')\hat{\mathbf{V}}_{p_2,2}^{-1} N^{-2}\mathbf{X}^{\mathrm{T}} \left\{ n^{-1}\sum_{i=1}^{n} U_{ij}\tilde{\varepsilon}_{p_1}(j'/N) \right\}_{1\leq j\neq j' \leq N} \right\|_\infty \\ &= O\left\{ n^{-1/2}N^{-1/2}h_{\mathrm{s}_1}^{-1}h_{\mathrm{s}_2}^{1/2}h_*^{1/2}h_{\mathrm{s}_2}^{-2} \right\} = O\left\{ n^{-1/2}N^{-1/2}h_{\mathrm{s}_1}^{-1/2}h_{\mathrm{s}_2}^{-3/2} \right\} = o(1). \end{aligned}$$

Similarly, $\tilde{\xi}_{p_1}(j'/N) = \sum_{J=1-p_1}^{N_{s_1}} B_{J,p_1}(j'/N)\, s_J$, where

$$\{s_J\}_{J=1-p_1}^{N_{s_1}} = N^{-1}\hat{\mathbf{V}}_{p_1}^{-1}\mathbf{X}_1^{\mathrm{T}} \sum_{k=1}^{\infty} \bar{\xi}_{\cdot k}\boldsymbol{\phi}_k.$$

Assumption (A3) ensures that

$$\begin{aligned} & \left\| \mathbf{X}^{\mathrm{T}} \left\{ n^{-1}\sum_{i=1}^{n} U_{ij}\tilde{\xi}_{p_1}(j'/N) \right\}_{1\leq j\neq j'\leq N} \right\|_2^2 \\ \leq & \sum_{J,J'=1-p_2}^{N_{s_2}} \sum_{J''=1-p_1}^{N_{s_1}} \left[n^{-1}\sum_{i=1}^{n}\sum_{1\leq j\neq j'\leq N} U_{ij}\times B_{J'',p_1}(j'/N)\, B_{JJ',p_2}(j/N,j'/N) \right]^2 \\ & \times \sum_{J''=1-p_1}^{N_{s_1}} s_{J''}^2 \\ = & \; I\times III, \end{aligned}$$

where $III = \left\| N^{-1}\hat{\mathbf{V}}_{p_1}^{-1}\mathbf{X}_1^{\mathrm{T}} \sum_{k=1}^{\infty} \bar{\xi}_{\cdot k}\boldsymbol{\phi}_k \right\|_2^2$. Note that

$$N^{-1}\mathbf{X}_1^{\mathrm{T}}\sum_{k=1}^{\infty}\bar{\xi}_{\cdot k}\boldsymbol{\phi}_k = \left\{ \sum_{k=1}^{\infty}\bar{\xi}_{\cdot k}N^{-1}\sum_{j=1}^{N} B_{J,p_1}(j/N)\,\phi_k(j/N) \right\}_{J=1-p_1}^{N_{s_1}},$$

and

$$\begin{aligned} E\left[\sum_{k=1}^{\infty}\bar{\xi}_{\cdot k}N^{-1}\sum_{j=1}^{N} B_{J,p_1}(j/N)\,\phi_k(j/N) \right]^2 & \leq Ch_{\mathrm{s}_1}^2 \sup_{x\in[0,1]}\sum_{k=1}^{\infty}\phi_k^2(x)\, E\left(\bar{\xi}_{\cdot k}\right)^2 \\ & = O\left(n^{-1}h_{\mathrm{s}_1}^2\right), \end{aligned}$$

hence

$$III \leq C_{p_1}\left\| N^{-1}\mathbf{X}_1^{\mathrm{T}}\sum_{k=1}^{\infty}\bar{\xi}_{\cdot k}\boldsymbol{\phi}_k \right\|_2^2 h_{\mathrm{s}_1}^{-2} = O_p\left\{ (nh_{\mathrm{s}_1})^{-1} \right\}$$

and

$$n^{1/2} \sup_{x,x'\in[0,1]^2} \left\| \boldsymbol{B}_{p_2}^{\mathrm{T}}(x,x')\hat{\mathbf{V}}_{p_2,2}^{-1} N^{-2}\mathbf{X}^{\mathrm{T}} \times \left\{ n^{-1}\sum_{i=1}^{n} U_{ij}\tilde{\xi}_{p_1}(j'/N) \right\}_{1\le j\ne j'\le N} \right\|_{\infty}$$
$$= O\left\{ n^{-1/2} h_{s_1}^{-1/2} h_{s_2}^{1/2} h_*^{1/2} h_{s_2}^{-2} \right\} = O(n^{-1/2}h_{s_2}^{-3/2}) = o(1).$$

Next one obtains that

$$E\left\{ n^{-1}N^{-2}\sum_{i=1}^{n}\sum_{1\le j\ne j'\le N} U_{ij}B_{J,p_2}(j/N)\,B_{J',p_2}(j'/N)\,(m-\tilde{m}_{p_1})(j'N) \right\}^2$$
$$= n^{-1}N^{-4}\sum_{j,j''=1}^{N} E\left(U_{1j}U_{1j''}\right) B_{J,p_2}(j/N)\,B_{J,p_2}(j''/N)$$
$$\times \sum_{j'\ne j}^{N}\sum_{j'''\ne j''}^{N} B_{J',p_2}(j'/N)\,B_{J',p_1}(j'''/N)$$
$$\times (m-\tilde{m}_{p_1})(j'/N)\,(m-\tilde{m}_{p_1})(j'''/N)$$
$$\le C(G,\sigma^2) h_{s_1}^{2p_1} n^{-1}N^{-4}\left(Nh_{s_2}\right)^4 = C(G,\sigma^2)h_{s_1}^{2p_1}n^{-1}h_{s_2}^4.$$

Therefore,

$$\sup_{x,x'\in[0,1]^2}\left\| \boldsymbol{B}_{p_2}^{\mathrm{T}}(x,x')\hat{\mathbf{V}}_{p_2,2}^{-1}N^{-2}\mathbf{X}^{\mathrm{T}}\left\{ n^{-1}\sum_{i=1}^{n}U_{ij}\left(m(j'/N)-\tilde{m}_{p_1}(j'/N)\right)\right\}_{1\le j\ne j'\le N}\right\|_{\infty}$$
$$= O(n^{-1/2}h_{s_2}^{-2}h_{s_1}^{p_1}h_{s_2}) = O(n^{-1/2}h_{s_2}^{-1}h_{s_1}^{p_1}) = o\left(n^{-1/2}\right).$$

Finally, we derive the upper bound of

$$\sup_{x,x'\in[0,1]^2}\left\| \boldsymbol{B}_{p_2}^{\mathrm{T}}(x,x')\hat{\mathbf{V}}_{p_2,2}^{-1}N^{-2}\mathbf{X}^{\mathrm{T}}\left(\mathbf{m}-\hat{\mathbf{m}}_{p_1}\right)^{\otimes 2}\right\|_{\infty},$$

where

$$\left(\mathbf{m}-\hat{\mathbf{m}}_{p_1}\right)^{\otimes 2} = \left\{ (m-\hat{m}_{p_1})(j/N)\,(m-\hat{m}_{p_1})(j'/N)\right\}_{1\le j\ne j'\le N}.$$

In order to apply Lemma 4, one needs to find the upper bound of $\left\|\left(\mathbf{m}-\hat{\mathbf{m}}_{p_1}\right)^{\otimes 2}\right\|_{\infty}$. Using the similar proof as Lemma A.8 in Wang and Yang (2009) and Assumption (A3), one has $\sup_{x\in[0,1]}|\tilde{\varepsilon}_{p_1}(x)|+\sup_{x\in[0,1]}\left|\tilde{\xi}_{p_1}(x)\right|=o\left(n^{-1/2}\right)$. Therefore,

$$\begin{aligned}
&\sup_{(x,x')\in[0,1]^2}\{m(x)-\hat{m}_{p_1}(x)\}\{m(x')-\hat{m}_{p_1}(x')\}\\
&\leq\left[\sup_{x\in[0,1]}(m(x)-\hat{m}_{p_1}(x))\right]^2\\
&\leq\left\{\sup_{x\in[0,1]}|m(x)-\tilde{m}_{p_1}(x)|+\sup_{x\in[0,1]}|\tilde{\varepsilon}_{p_1}(x)|+\sup_{x\in[0,1]}\left|\tilde{\xi}_{p_1}(x)\right|\right\}^2\\
&\leq\left[O\left(h_{s_1}^{p_1}+n^{-1/2}\right)\right]^2=O\left(h_{s_1}^{2p_1}+n^{-1}+h_{s_1}^{p_1}n^{-1/2}\right)=o\left(n^{-1/2}\right).
\end{aligned}$$

Hence $\left\|(\mathbf{m}-\hat{\mathbf{m}}_{p_1})^{\otimes 2}\right\|_\infty=o\left(n^{-1/2}\right)$. Hence, the proposition has been proved.

S4 Proof of Theorem ??

First, denoted by

$$\begin{aligned}
\mathcal{M}(x,x') &= E\{\eta_1^2(x)\eta_1^2(x')\}\\
&= E\left\{\sum_{k=1}^{\kappa}\xi_{1k}\phi_k(x)\right\}^2\left\{\sum_{k'=1}^{\kappa}\xi_{1k'}\phi_{k'}(x')\right\}^2\\
&= \sum_{k_1}^{\kappa}\sum_{k_2}^{\kappa}\sum_{k_3}^{\kappa}\sum_{k_4}^{\kappa}E(\xi_{1k_1}\xi_{1k_2}\xi_{1k_3}\xi_{1k_4})\phi_{k_1}(x)\phi_{k_2}(x)\phi_{k_3}(x')\phi_{k_4}(x')\\
&= \sum_{k_1}^{\kappa}\sum_{k_3\neq k_1}^{\kappa}\phi_{k_1}^2(x)\phi_{k_3}^2(x')+2\sum_{k_1}^{\kappa}\sum_{k_2\neq k_1}^{\kappa}\phi_{k_1}(x)\phi_{k_2}(x)\phi_{k_1}(x')\phi_{k_2}(x')\\
&+ \sum_{k}^{\kappa}E(\xi_{1k}^4)\phi_k^2(x)\phi_k^2(x')\\
&= \sum_{k_1}^{\kappa}\sum_{k_3}^{\kappa}\phi_{k_1}^2(x)\phi_{k_3}^2(x')+2\sum_{k_1}^{\kappa}\sum_{k_2}^{\kappa}\phi_{k_1}(x)\phi_{k_2}(x)\phi_{k_1}(x')\phi_{k_2}(x')\\
&+ \sum_{k}^{\kappa}E(\xi_{1k}^4-3)\phi_k^2(x)\phi_k^2(x')\\
&= G(x,x)G(x',x')+2G^2(x,x')+\sum_{k}^{\kappa}+E(\xi_{1k}^4-3)\phi_k^2(x)\phi_k^2(x')\\
&= V(x,x')+G^2(x,x')
\end{aligned}$$

For any $j \neq j'$, one has

$$\begin{aligned} E(U_{1j}^2 U_{1j'}^2) &= E\{(\eta_{1j} + \sigma(j/N)\varepsilon_{1j})^2 (\eta_{1j'} + \sigma(j'/N)\varepsilon_{1j'})^2\} \\ &= \mathcal{M}(j/N, j'/N) + G(j/N, j/N)\sigma^2(j'/N) + G(j'/N, j'/N)\sigma^2(j/N) \\ &+ \sigma^2(j/N)\sigma^2(j'/N) \end{aligned}$$

Using the similar arguments as in Theorem 1, we can show the tensor product spline estimator $\hat{\mathcal{M}}_{p_1,p_2}(x, x')$ is an uniformly consistent estimator of $\mathcal{M}(x, x')$. Therefore, $\hat{V}_{p_1,p_2}(x, x')$ is also an uniformly consistent estimator of $V(x, x')$.

References

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