Supplementary Material to "Prediction Interval for Autoregressive Time Series via Oracally Efficient Estimation of Multi-Step Ahead Innovation Distribution Function"

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Abstract

The supporting information provides the proof of Lemma A.5 in the Appendix and some additional simulation results as described in Section 4 of the main text.

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Lemma A.5. *Under Assumptions (A1)-(A5), for any $k \leq r,s,v,w \leq k+p-1$, as $n \to \infty$,*

$$\sup_{z\in\mathbb{R}}\left|n^{-1}\sum_{t=k}^{n}K_h\left(z-Z_t^{[k]}\right)X_{t-r}\right| = \mathcal{O}_p\left(n^{-1/2}h^{-1/2}\log n\right), \tag{A.2}$$

$$\sup_{z\in\mathbb{R}}\left|n^{-1}\sum_{t=k}^{n}K_h^{'}\left(z-Z_t^{[k]}\right)X_{t-r}X_{t-s}\right| = \mathcal{O}_p\left(1\right), \tag{A.3}$$

$$\sup_{z\in\mathbb{R}}\left|n^{-1}\sum_{t=k}^{n}K_h^{''}\left(z-Z_t^{[k]}\right)X_{t-r}X_{t-s}X_{t-v}\right| = \mathcal{O}_p\left(1\right), \tag{A.4}$$

$$n^{-1}\sum_{t=k}^{n}\left|X_{t-r}X_{t-s}X_{t-v}X_{t-w}\right| = \mathcal{O}_p\left(1\right). \tag{A.5}$$

Proof According to Assumption (A3), $X_t = \sum_{j=0}^{\infty}\psi_j Z_{t-j}$, while $Z_t^{[k]} = \sum_{j=0}^{k-1}\psi_j Z_{t-j}$ in (2.4), so $Z_t^{[k]}$ and $X_{t-r}, r \geq k$ are independent. We decompose X_{t-r} into tail and truncated parts as $X_{t-r} = X_{t-r,1}^{D_n} + X_{t-r,2}^{D_n} + X_{t-r,3}^{D_n}$, where $X_{t-r,1}^{D_n} = X_{t-r}I\left\{|X_{t-r}| > D_n\right\}$,

$$\begin{aligned} X_{t-r,2}^{D_n} &= \mathbb{E}\left[X_{t-r}I\left\{|X_{t-r}| \leq D_n\right\}|Z_t^{[k]}\right] = \mathbb{E}X_{t-r}I\left\{|X_{t-r}| \leq D_n\right\}, \\ X_{t-r,3}^{D_n} &= X_{t-r}I\left\{|X_{t-r}| \leq D_n\right\} - X_{t-r,2}^{D_n} \\ &= X_{t-r}I\left\{|X_{t-r}| \leq D_n\right\} - \mathbb{E}X_{t-r}I\left\{|X_{t-r}| \leq D_n\right\}, \end{aligned}$$

and correspondingly denote $\xi_t = \xi_t(z) = n^{-1}K_h\left(z-Z_t^{[k]}\right)X_{t-r} = \xi_{t,1}+\xi_{t,2}+\xi_{t,3}$ in which

$$\xi_{t,l} = \xi_{t,l}(z) = n^{-1}K_h\left(z-Z_t^{[k]}\right)X_{t-r,l}^{D_n}, l = 1,2,3.$$

Based on Lemma A.4 and (2.3),

$$\sum\nolimits_{n=1}^{\infty}\mathrm{P}\left(|X_{n-r}| > D_n\right) \leq \sum\nolimits_{n=1}^{\infty}D_n^{-(2+\eta)}\mathbb{E}\left(|X_{n-r}|^{2+\eta}\right) < \infty.$$

According to Borel-Cantelli Lemma, one has

$$\mathrm{P}\left\{\omega\left|\exists N_1(\omega), |X_{n-r}| \leq D_n, \forall n > N_1(\omega)\right.\right\} = 1.$$

Since $\{D_n\} = \{n^a\}$ is increasing, one concludes that

$$\mathrm{P}\left\{\omega\left|\exists N(\omega), |X_{t-r}| \leq D_n, t = k,\ldots,n, n > N(\omega)\right.\right\} = 1. \tag{S.1}$$

It follows from (S.1) that $\sum_{t=k}^{n}\xi_{t,1} = U_{a.s.}\left(n^{-\kappa}\right)$ for any $\kappa > 0$.

The truncation mean satisfies $\left|X_{t-r,2}^{D_n}\right| \leq D_n^{-(1+\eta)}\mathbb{E}\left|X_{t-r}\right|^{2+\eta}$, and since $D_n^{-(1+\eta)}n^{1/2}h^{1/2} \to 0$ according to Lemma A.4, classic kernel smoothing theory yields $\sum_{t=k}^{n}\xi_{t,2} = U_p\left(D_n^{-(1+\eta)}\right) = u_p\left(n^{-1/2}h^{-1/2}\right)$.

To prove $\sum_{t=k}^{n}\xi_{t,3} = \mathcal{O}_p\left(n^{-1/2}h^{-1/2}\log n\right)$, Lemma A.2 will be applied. Note that

$$\mathbb{E}\left(X_{t-r,3}^{D_n}\,\middle|\,Z_t^{[k]}\right) = 0, \mathrm{Var}\left(X_{t-r,3}^{D_n}\,\middle|\,Z_t^{[k]}\right) = \mathbb{E}\left(\left|X_{t-r}\right|^2\right) + U_p\left(D_n^{-2(1+\eta)} + D_n^{-\eta}\right).$$

Clearly, $\mathbb{E}\xi_{t,3} = 0$ and

$$\begin{aligned} m_2^2 = \mathbb{E}\left|\xi_{t,3}\right|^2 &= n^{-2}\mathbb{E}K_h^2\left(z - Z_t^{[k]}\right)\mathbb{E}\left(X_{t-r,3}^{D_n}\right)^2 \\ &= \mathbb{E}\left(\left|X_{t-r}\right|^2\right)f^{[k]}(z)\,n^{-2}h^{-1}\int K^2(\nu)\,d\nu\left\{1 + u(1)\right\}, \end{aligned}$$

so for n large enough,

$$m_2^2 \leq 2\mathbb{E}\left(\left|X_{t-r}\right|^2\right)\left\|f^{[k]}\right\|_\infty n^{-2}h^{-1}\left\|K\right\|_2^2 = Cn^{-2}h^{-1}\left\|K\right\|_2^2. \tag{S.2}$$

For any integer $\mu \geq 3$,

$$\begin{aligned} \mathbb{E}\left|\xi_{t,3}\right|^\mu &= n^{-\mu}\mathbb{E}\left|K_h\left(z - Z_t^{[k]}\right)\left\{X_{t-r}I\left\{\left|X_{t-r}\right| \leq D_n\right\} - X_{t-r,2}^{D_n}\right\}\right|^\mu \\ &\leq n^{-\mu}\mathbb{E}\left|K_h\left(z - Z_t^{[k]}\right)\right|^\mu (2D_n)^\mu \leq 2\left\|f^{[k]}\right\|_\infty h\left(2D_n n^{-1}h^{-1}\left\|K\right\|_\infty\right)^\mu. \end{aligned} \tag{S.3}$$

Note that,

$$\begin{aligned} \left|\xi_{t,3}\right| &= \left|n^{-1}K_h\left(z - Z_t^{[k]}\right)\left[X_{t-r}I\left\{\left|X_{t-r}\right| \leq D_n\right\} - X_{t-r,2}^{D_n}\right]\right| \\ &\leq n^{-1}h^{-1}\left\|K\right\|_\infty(2D_n), \end{aligned}$$

then,

$$\begin{aligned} \mathbb{E}\left|\xi_{t,3}\right|^\mu &= \mathbb{E}\left|\xi_{t,3}\right|^{\mu-2}\left|\xi_{t,3}\right|^2 \leq \left(2D_n n^{-1}h^{-1}\left\|K\right\|_\infty\right)^{\mu-2}\mathbb{E}\left|\xi_{t,3}\right|^2 \\ &\leq \mu!\left(2D_n n^{-1}h^{-1}\left\|K\right\|_\infty\right)^{\mu-2}\mathbb{E}\left|\xi_{t,3}\right|^2, \mu \geq 3. \end{aligned}$$

So Cramér's condition as in Lemma A.2 is fulfilled with $2D_n n^{-1}h^{-1}\left\|K\right\|_\infty$ as the growth constant c. Applying Lemma A.2 with $\mu = 3$, let $\varepsilon_n = C_\varepsilon n^{-3/2}h^{-1/2}\log n$ for some $C_\varepsilon > 0$, and $q = \left[c_q\left(\log n\right)^{-1}n\right] + 1$, for some $c_q > 0$, then $q^{-1}n \sim c_q^{-1}\log n$. According to (S.3), one has

$$m_3 = \max_{k \leq t \leq n}\left\|\xi_{t,3}\right\|_3 \leq (2\left\|f^{[k]}\right\|_\infty h)^{1/3}2D_n\left(nh\right)^{-1}\left\|K\right\|_\infty = Cn^{a-1}h^{-2/3},$$

thus $m_3^{6/7} \leq (Cn^{3a-3}h^{-2})^{2/7} = Cn^{6(a-1)/7}h^{-4/7}$. While by (S.2), one has

$$2\left\{1+\varepsilon_n^2/\left(25m_2^2+5c\varepsilon_n\right)\right\} \sim 2\left\{1+Cn^{-1}(\log n)^2\right\},$$

so $a_1 \leq 3c_q^{-1}\log n$ for large n, and as C_ε, n large enough,

$$\begin{aligned} \frac{q\varepsilon_n^2}{25m_2^2+5c\varepsilon_n} &\geq \frac{c_q C_\varepsilon^2 \log n}{25C\left\|K\right\|_2^2+10D_n\left\|K\right\|_\infty C_\varepsilon n^{-1/2}h^{-1/2}\log n} \\ &\geq \frac{c_q C_\varepsilon^2 \log n}{50C\left\|K\right\|_2^2} \geq (9+C_2)\log n, \text{ for some } C_2. \end{aligned}$$

Lastly, $a_2(3) \leq 11n\left\{1+5Cn^{(12a+9)/14}h^{-1/14}\left(C_\varepsilon \log n\right)^{-1}\right\} \leq Cn^3$. Notice that there exist positive constants $C_\rho > 0$ and $\rho \in (0,1)$ such that $\alpha(t) \leq C_\rho \rho^t$. By taking sufficiently small c_q, for $\mu = 3$,

$$\begin{aligned} \alpha\left(\left[(n-k)/\left(q+1\right)\right]\right)^{2\mu/(2\mu+1)} &\leq \alpha\left(\left[\log n/2c_q\right]\right)^{6/7} \\ &\leq C_\rho^{6/7}\rho^{6/7[\log n/2c_q]} \leq Cn^{-11-C_2}. \end{aligned}$$

Putting all together, for some constants $C_\varepsilon, c_q > 0$ and large n,

$$\begin{aligned} \mathrm{P}\left\{\left|\sum_{t=k}^n \xi_{t,3}\right| > (n-k)\varepsilon_n\right\} &\leq 3c_q^{-1}\left(\log n\right)\exp\left\{-\left(9+C_2\right)\log n\right\} + Cn^3 n^{-11-C_2} \\ &\leq Cn^{-8-C_2}. \end{aligned}$$

According to Lemma A.2, one obtains

$$\mathrm{P}\left\{\left|\sum_{t=k}^n \xi_{t,3}\right| > C_\varepsilon\left(nh\right)^{-1/2}\log n\right\} \leq \mathrm{P}\left\{\left|\sum_{t=k}^n \xi_{t,3}\right| > (n-k)\varepsilon_n\right\} \leq Cn^{-8-C_2}.$$

To bound the truncated sum uniformly for all $z \in [-a_n, a_n]$, where the sequence $a_n > 0$, $a_n \to \infty$, $a_n \leq C_1 n^{C_2}$ for some $C_1, C_2 > 0$, we discrete by equally spaced $-a_n = z_0 < z_1 < \cdots < z_N < a_n$ with $N = n^{5+[C_2]}$,

$$\begin{aligned} &\mathrm{P}\left\{\max_{j=0,1,\ldots,N}\left|\sum_{t=k}^n \xi_{t,3}\left(z_j\right)\right| > C_\varepsilon\left(nh\right)^{-1/2}\log n\right\} \\ \leq &\sum_{j=0}^N \mathrm{P}\left\{\left|\sum_{t=k}^n \xi_{t,3}\left(z_j\right)\right| > C_\varepsilon\left(nh\right)^{-1/2}\log n\right\} \leq Cn^{-3}. \end{aligned}$$

So Borel-Cantelli Lemma implies that

$$\max_{j=0,1,\ldots,N}\left|\sum\nolimits_{t=k}^n \xi_{t,3}\left(z_j\right)\right| = \mathcal{O}_{a.s.}\left(n^{-1/2}h^{-1/2}\log n\right).$$

Notice that

$$\begin{aligned}
& \sup_{z\in[-a_n,a_n]}\left|\sum\nolimits_{t=k}^{n}\xi_{t,3}\left(z\right)\right| \\
\leq\ & \max_{j=0,1,\ldots,N}\left|\sum_{t=k}^{n}\xi_{t,3}\left(z_j\right)\right| + \max_{j=0,1,\ldots,N-1}\sup_{z\in[z_j,z_{j+1}]}\left|\sum_{t=k}^{n}\xi_{t,3}\left(z_j\right)-\sum_{t=k}^{n}\xi_{t,3}\left(z\right)\right| \\
=\ & \mathcal{O}_{a.s.}\left(n^{-1/2}h^{-1/2}\log n\right)+\mathcal{O}_p\left(n^{-(5+[C_2])}4a_nh^{-2}D_n\left\|K'\right\|_\infty\right) \\
=\ & \mathcal{O}_p\left(n^{-1/2}h^{-1/2}\log n\right).
\end{aligned}$$

Combining the three parts together, one obtains

$$\sup_{z\in[-a_n,a_n]}\left|n^{-1}\sum\nolimits_{t=k}^{n}K_h\left(z-Z_t^{[k]}\right)X_{t-r}\right| = \mathcal{O}_p\left(n^{-1/2}h^{-1/2}\log n\right). \tag{S.4}$$

In the following, we need to prove

$$\sup_{|z|\geq a_n}\left|n^{-1}\sum\nolimits_{t=k}^{n}K_h\left(z-Z_t^{[k]}\right)X_{t-r}\right| = \mathcal{O}_p(n^{-1}). \tag{S.5}$$

For $z\in(-\infty,-a_n)$, one has

$$\begin{aligned}
& \sup_{z\in(-\infty,-a_n)}\left|n^{-1}\sum\nolimits_{t=k}^{n}K_h\left(z-Z_t^{[k]}\right)X_{t-r}\right| \qquad (S.6)\\
\leq\ & n^{-1}\sum\nolimits_{t=k}^{n}I\left\{Z_t^{[k]}\in(-\infty,-a_n+h]\right\}h^{-1}\left\|K\right\|_\infty \max\nolimits_{1\leq t\leq n}\left|X_t\right|.
\end{aligned}$$

Denote $A=n^{-1}\sum_{t=k}^{n}I\left\{Z_t^{[k]}\in(-\infty,-a_n+h]\right\}$, which can be written as

$$A=\mathrm{P}\left(Z_t^{[k]}\leq -a_n+h\right)+\sum\nolimits_{t=k}^{n}\xi_{tn},$$

where $\xi_{tn}=n^{-1}\left[I\left\{Z_t^{[k]}\leq -a_n+h\right\}-\mathrm{P}\left(Z_t^{[k]}\leq -a_n+h\right)\right]$. The first part of A, $\mathrm{P}\left(Z_t^{[k]}\leq -a_n+h\right)\leq \mathbb{E}\left|Z_t^{[k]}\right|^{2+\eta}(a_n-h)^{-2-\eta}\leq \mathbb{E}\left|Z_t^{[k]}\right|^{2+\eta}n^{-s(2+\eta)}$ by the selection of $a_n=h+n^s$. Consider the second part of A, $\mathbb{E}\xi_{tn}=0$,

$$\mathbb{E}\xi_{tn}^2=\mathrm{P}\left(Z_t^{[k]}\leq -a_n+h\right)\left\{1-\mathrm{P}\left(Z_t^{[k]}\leq -a_n+h\right)\right\}/n^2\leq \mathbb{E}\left|Z_t^{[k]}\right|^{2+\eta}n^{-s(2+\eta)-2}.$$

Therefore $\sum_{t=k}^{n}\xi_{tn}=\mathcal{O}_p\left(n^{-\{s(2+\eta)+1\}/2}\right)$, and $A=\mathcal{O}_p\left(n^{-\{s(2+\eta)+1\}/2}\right)$. Using this bound on A, (S.6) and Lemma A.3, one obtains that

$$\begin{aligned}
& \sup_{z\in(-\infty,-a_n)}\left|n^{-1}\sum\nolimits_{t=k}^{n}K_h\left(z-Z_t^{[k]}\right)X_{t-r}\right| \\
=\ & \mathcal{O}_p\left(n^{-\{s(2+\eta)+1\}/2+\gamma}h^{-1}\right)=\mathcal{O}_p\left(n^{-\{s(2+\eta)+1\}/2+\gamma+3/8}\right)=\mathcal{O}_p\left(n^{-1}\right)
\end{aligned}$$

as the choice of $s>(7/4+2\gamma)(2+\eta)^{-1}$ entails that $-\{s(2+\eta)+1\}/2+\gamma+3/8<-1$. The proof for $z\in(a_n,\infty)$ is similar. Hence, (S.4) and (S.5) together imply (A.2). (A.3) and (A.4) are obtained similarly. The proof of (A.5) is similar to that of Lemma A.6 in Wang et al. (2014). □

Table S.1: The 95% PIs' coverage frequencies of X_{n+3} for AR(1) in Case 1 (Normal) over 1000 replications.

ϕ	n	infeasible	oracle	normal	empirical	bootstrap
-0.8	50	0.960	0.925	0.938	0.910	0.928
	100	0.945	0.936	0.940	0.933	0.930
	500	0.950	0.950	0.946	0.949	0.953
	1000	0.951	0.950	0.950	0.948	0.945
-0.4	50	0.96	0.946	0.954	0.932	0.933
	100	0.945	0.938	0.944	0.934	0.942
	500	0.941	0.942	0.942	0.942	0.954
	1000	0.952	0.945	0.948	0.946	0.948
-0.2	50	0.965	0.941	0.962	0.930	0.942
	100	0.947	0.940	0.948	0.937	0.940
	500	0.943	0.941	0.943	0.939	0.947
	1000	0.943	0.940	0.939	0.940	0.945
0.2	50	0.951	0.926	0.940	0.908	0.944
	100	0.954	0.951	0.952	0.946	0.940
	500	0.943	0.944	0.944	0.942	0.942
	1000	0.944	0.945	0.944	0.946	0.957
0.4	50	0.948	0.908	0.931	0.897	0.939
	100	0.965	0.950	0.953	0.945	0.936
	500	0.947	0.939	0.946	0.940	0.935
	1000	0.953	0.952	0.954	0.952	0.960
0.8	50	0.953	0.893	0.916	0.880	0.895
	100	0.958	0.934	0.946	0.924	0.922
	500	0.951	0.947	0.948	0.947	0.941
	1000	0.953	0.953	0.954	0.953	0.957

Table S.2: The 95% PIs' coverage frequencies of X_{n+3} for AR(2) in Case 2 (Kurtotic) over 1000 replications.

(ϕ_1, ϕ_2)	n	infeasible	oracle	normal	empirical	bootstrap
$(-0.8, -0.4)$	50	0.960	0.936	0.939	0.928	0.938
	100	0.955	0.940	0.934	0.934	0.936
	500	0.960	0.950	0.935	0.950	0.939
	1000	0.949	0.946	0.937	0.945	0.946
$(0.8, -0.4)$	50	0.957	0.915	0.927	0.899	0.933
	100	0.947	0.931	0.927	0.927	0.944
	500	0.959	0.959	0.944	0.958	0.948
	1000	0.943	0.941	0.933	0.943	0.940
$(0.2, -0.1)$	50	0.956	0.918	0.925	0.910	0.955
	100	0.958	0.944	0.934	0.941	0.941
	500	0.951	0.946	0.927	0.946	0.950
	1000	0.945	0.948	0.933	0.948	0.952
$(-0.2, 0.1)$	50	0.962	0.929	0.928	0.921	0.948
	100	0.956	0.939	0.92	0.938	0.942
	500	0.952	0.948	0.927	0.948	0.944
	1000	0.945	0.945	0.926	0.947	0.954
$(0.1, -0.05)$	50	0.958	0.921	0.926	0.914	0.956
	100	0.953	0.940	0.930	0.940	0.939
	500	0.951	0.944	0.924	0.945	0.947
	1000	0.948	0.942	0.932	0.943	0.951
$(-0.1, 0.05)$	50	0.961	0.924	0.924	0.913	0.950
	100	0.952	0.940	0.924	0.937	0.942
	500	0.952	0.945	0.927	0.945	0.951
	1000	0.944	0.945	0.920	0.946	0.950

Table S.3: The 95% PIs' coverage frequencies of X_{n+3} for AR(2) in Case 3 (Bimodal) over 1000 replications.

(ϕ_1, ϕ_2)	n	infeasible	oracle	normal	empirical	bootstrap
$(-0.8, -0.4)$	50	0.950	0.931	0.967	0.908	0.930
	100	0.961	0.943	0.978	0.932	0.941
	500	0.949	0.954	0.976	0.952	0.950
	1000	0.951	0.950	0.982	0.948	0.961
$(0.8, -0.4)$	50	0.948	0.918	0.958	0.894	0.922
	100	0.948	0.932	0.959	0.924	0.936
	500	0.950	0.943	0.980	0.943	0.955
	1000	0.938	0.937	0.975	0.936	0.953
$(0.2, -0.1)$	50	0.947	0.948	0.996	0.905	0.965
	100	0.951	0.949	0.995	0.931	0.959
	500	0.939	0.937	0.996	0.934	0.946
	1000	0.947	0.949	0.997	0.946	0.952
$(-0.2, 0.1)$	50	0.946	0.954	0.994	0.924	0.962
	100	0.955	0.937	0.996	0.938	0.960
	500	0.941	0.944	0.995	0.939	0.946
	1000	0.947	0.945	0.995	0.943	0.956
$(0.1, -0.05)$	50	0.951	0.953	0.998	0.916	0.976
	100	0.945	0.948	0.998	0.930	0.967
	500	0.942	0.940	1.000	0.933	0.949
	1000	0.948	0.952	0.997	0.945	0.959
$(-0.1, 0.05)$	50	0.940	0.949	0.998	0.919	0.977
	100	0.950	0.957	1.000	0.944	0.971
	500	0.940	0.941	1.000	0.936	0.946
	1000	0.943	0.945	0.995	0.943	0.965