Prediction Interval for Autoregressive Time Series via Oracally Efficient Estimation of Multi-Step Ahead Innovation Distribution Function

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Abstract

Kernel distribution estimator (KDE) is proposed for multi-step ahead prediction error distribution of autoregressive time series, based on prediction residuals. Under general assumptions, the KDE is proved to be oracally efficient as the infeasible KDE and the empirical cdf based on unobserved prediction errors. Quantile estimator is obtained from the oracally efficient KDE and prediction interval for multi-step ahead future observation is constructed using the estimated quantiles and shown to achieve asymptotically the nominal confidence levels. Simulation examples corroborate the asymptotic theory.
Keywords: AR(p), Non-Gaussian distribution, Prediction interval, Residuals, Yule-Walker estimator.

1 INTRODUCTION

Forecasting occupies a central place in the study of time series, with wide applications to economics, finance and other disciplines, see data examples from Brockwell and

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Davis (1991). Consider a causal AR(p) time series $\{X_t\}_{t=-\infty}^{+\infty}$ which is a stochastic process that satisfies

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + Z_t,$$

in which the unobserved $\{Z_t\}_{t=-\infty}^{+\infty}$ are iid innovations, $\mathbb{E}Z_t = 0$, $\mathbb{E}Z_t^2 = \sigma^2$, with pdf $f(z)$ and cdf $F(z) = \int_{-\infty}^{z} f(u)\,du$. A time series data $\{X_t\}_{t=1-p}^{n}$ consists of a length $n+p$ realization of $\{X_t\}_{t=-\infty}^{+\infty}$, and the k-step ahead linear predictor $\tilde{X}_{n+k}^{[k]}$ for $X_{n+k}, k \geq 1$ based on $\{X_t\}_{t=1-p}^{n}$, is defined recursively by

$$\tilde{X}_{n+k}^{[k]} = \phi_1 \tilde{X}_{n+k-1}^{[k-1]} + \cdots + \phi_p \tilde{X}_{n+k-p}^{[k-p]}, \tag{1.1}$$

and satisfies

$$\tilde{X}_{n+k}^{[k]} = \phi_1^{[k]} X_n + \cdots + \phi_p^{[k]} X_{n-p+1}, \tag{1.2}$$

in which the coefficient vector $\boldsymbol{\phi}^{[k]} = \left(\phi_1^{[k]}, \ldots, \phi_p^{[k]}\right)^{\mathsf{T}}$ is a polynomial function g_k of $\boldsymbol{\phi} = \left(\phi_1, \ldots, \phi_p\right)^{\mathsf{T}}$: $\boldsymbol{\phi}^{[k]} = g_k(\boldsymbol{\phi})$, with g_k defined recursively by repeated applications of (1.1). One naturally would ask if there exists a data-driven prediction interval (PI) for X_{n+k} based on $\{X_t\}_{t=1-p}^{n}$.

A $100(1-\alpha)\%$ k-step ahead *normal PI* for X_{n+k} is given in Section 5.4 of Brockwell and Davis (1991) as $\tilde{X}_{n+k}^{[k]} \pm \Phi^{-1}(1-\alpha/2)\,\sigma_{[k]}$, where $\sigma_{[k]}$ is the standard deviation of the k-step ahead prediction errors $Z_{n+k}^{[k]} = X_{n+k} - \tilde{X}_{n+k}^{[k]}$ and $\Phi^{-1}(1-\alpha/2)$ the $(1-\alpha/2)$-quantile of the standard normal distribution. Clearly, the validity of this PI presumes the conditional distribution of $Z_{n+k}^{[k]}$ being $N\left(0, \sigma_{[k]}^2\right)$, so the question remains unaddressed how this normal PI performs if prediction errors $Z_{n+k}^{[k]}$ are significantly non-Gaussian.

If one possessed knowledge of the k-step ahead prediction error distribution $F^{[k]}(z)$, and obtained its $\alpha/2$-th and $(1-\alpha/2)$-th quantiles $q_{\alpha/2}^{[k]}$ and $q_{1-\alpha/2}^{[k]}$, then the following

$$\mathrm{P}\left(X_{n+k} \in \left[\tilde{X}_{n+k}^{[k]} + q_{\alpha/2}^{[k]}, \tilde{X}_{n+k}^{[k]} + q_{1-\alpha/2}^{[k]}\right]\right) = \mathrm{P}\left(Z_{n+k}^{[k]} \in \left[q_{\alpha/2}^{[k]}, q_{1-\alpha/2}^{[k]}\right]\right) = 1-\alpha$$

would lead to the following $100(1-\alpha)\%$ *infeasible PI* for X_{n+k} which achieves the nominal confidence level:

$$\left[\tilde{X}_{n+k}^{[k]} + q_{\alpha/2}^{[k]}, \tilde{X}_{n+k}^{[k]} + q_{1-\alpha/2}^{[k]}\right]. \tag{1.3}$$

This PI is termed "infeasible" as it contains unknown quantities $q_{\alpha/2}^{[k]}, q_{1-\alpha/2}^{[k]}$ and unknown coefficients $\phi_1^{[k]}, \ldots, \phi_p^{[k]}$. It does make one ask the question: What if one substitutes $q_{\alpha/2}^{[k]}, q_{1-\alpha/2}^{[k]}, \phi_1^{[k]}, \ldots, \phi_p^{[k]}$ by some "good" estimates and reconstructs a different PI?

Chapter 8 of Brockwell and Davis (1991) provides the Yule-Walker estimator $\hat{\boldsymbol{\phi}} = \left(\hat{\phi}_1, \ldots, \hat{\phi}_p\right)^\mathsf{T}$ of $\boldsymbol{\phi}$, one then obtains plug-in estimate $\hat{\boldsymbol{\phi}}^{[k]} = \left(\hat{\phi}_1^{[k]}, \ldots, \hat{\phi}_p^{[k]}\right)^\mathsf{T} = g_k\left(\hat{\boldsymbol{\phi}}\right)$ of $\boldsymbol{\phi}^{[k]} = g_k\left(\boldsymbol{\phi}\right)$. Define $\hat{X}_{n+k}^{[k]}$ as the data version of the linear predictor $\tilde{X}_{n+k}^{[k]}$:

$$\hat{X}_{n+k}^{[k]} = \hat{\phi}_1^{[k]} X_n + \cdots + \hat{\phi}_p^{[k]} X_{n-p+1}, \tag{1.4}$$

and $\hat{Z}_{n+k}^{[k]} = X_{n+k} - \hat{X}_{n+k}^{[k]}$ as the k-step ahead prediction residuals. We further propose an oracle estimator $\hat{q}_{n,\alpha}^{[k]} = \left(\hat{F}^{[k]}\right)^{-1}(\alpha) = \inf\left\{z : \hat{F}^{[k]}(z) \geq \alpha\right\}$ of $q_\alpha^{[k]}$ based on a two-step plug-in Kernel Distribution Estimator (KDE) $\hat{F}^{[k]}(z)$ of $F^{[k]}(z)$,

$$\hat{F}^{[k]}(z) = \int_{-\infty}^{z} \hat{f}^{[k]}(u)\, du = n^{-1} \sum_{t=k}^{n} \int_{-\infty}^{z} K_h\left(u - \hat{Z}_t^{[k]}\right) du, z \in \mathbb{R}, \tag{1.5}$$

in which K is a kernel function, with $K_h(u) = h^{-1} K(u/h)$, and $h = h_n > 0$ is called bandwidth, $\hat{Z}_t^{[k]} = X_t - \hat{X}_t^{[k]} = X_t - \hat{\phi}_1^{[k]} X_{t-k} - \cdots - \hat{\phi}_p^{[k]} X_{t-k-p+1}$, $k \leq t \leq n$ are prediction residuals. A $100(1-\alpha)\%$ *oracle PI* for X_{n+k} is then

$$\left[\hat{X}_{n+k}^{[k]} + \hat{q}_{n,\alpha/2}^{[k]}, \hat{X}_{n+k}^{[k]} + \hat{q}_{n,1-\alpha/2}^{[k]}\right]. \tag{1.6}$$

The adjective "*oracle*" means that asymptotically the PI achieves the same coverage probability as the infeasible PI.

One may wonder why we have not used the more natural empirical cdf $\hat{F}_n^{[k]}(z) = n^{-1}\sum_{t=k}^{n} I\left(\hat{Z}_t^{[k]} \leq z\right)$ of $\left\{\hat{Z}_t^{[k]}\right\}_{t=k}^{n}$ to estimate $F^{[k]}(z)$, and based on empirical quantile $\tilde{q}_{n,\alpha}^{[k]} = \inf\left\{z : \hat{F}_n^{[k]}(z) \geq \alpha\right\}$ to construct a $100(1-\alpha)\%$ *empirical PI* for X_{n+k} as

$$\left[\hat{X}_{n+k}^{[k]} + \tilde{q}_{n,\alpha/2}^{[k]}, \hat{X}_{n+k}^{[k]} + \tilde{q}_{n,1-\alpha/2}^{[k]}\right]. \tag{1.7}$$

In this paper, we have decided against using the empirical cdf $\hat{F}_n^{[k]}(z)$ for two reasons. The aesthetic one being that $F(z)$ is smooth while $\hat{F}_n^{[k]}(z)$ is not. More importantly, the definition of KDE $\hat{F}^{[k]}(z)$ in (1.5) allows for Taylor expansion and leads to simpler proof of oracle efficiency, as in Wang et al. (2014) for the special case of $k = 1$, when 1-step ahead prediction errors $\left\{Z_t^{[1]}\right\}_{t=1}^{n}$ are exactly iid innovations $\{Z_t\}_{t=1}^{n}$. In contrast, the same oracle efficiency was established in Bai (1994) between empirical cdf of residuals and of innovations, for the same case of $k = 1$, with much more tedious and longer proof. It is unclear to us if the techniques of Bai (1994) extend to the case of $k > 1$ when the k-step ahead prediction errors $\left\{Z_t^{[k]}\right\}_{t=k}^{n}$ in (2.4) are no longer independent as the $\left\{Z_t^{[1]}\right\}_{t=1}^{n}$. We have carried out simulation studies about empirical PI as a benchmark for performance, without theoretical support. To secure an oracally efficient estimator for $F^{[k]}(z)$ leading to consistent estimates of its quantiles, the KDE $\hat{F}^{[k]}(z)$ is a better choice from both theoretical and numerical aspects.

An interesting simulation comparison can be made using normal, infeasible, oracle, and empirical PIs described above, together with bootstrap PI proposed by Thombs and Schucany (1990). As a general principle, what one would like to have in an "ideal" PI are the following: First of all, it needs to be accurate, i.e., the probability of the unknown quantity being contained in the PI should be close to the prescribed nominal level $1 - \alpha$. Secondly, it should be informative, i.e., the interval being sufficiently narrow and therefore useful in locating the unknown quantity. With these in mind, consider an AR(2) time series with innovation distribution $F(z)$ being two normal mixtures studied in Marron and Wand (1992), for which we have coined the names "Bimodal" and "Kurtotic" due to their density shapes. Tables 1-3 list the empirical coverage frequencies from 1000 replications of the five PIs. Clearly, the normal PI's empirical coverage frequencies deviate considerably from the nominal coverage frequencies, which explains our motivation to propose the oracle PI as a robust device that works well regardless of the innovation distribution. Detailed comparison of the five PIs are stated in Section 4.

The paper is organized as follows. Section 2 gives the main asymptotically theoretical results, Section 3 the steps of implementation in detail. Section 4 reports the simulation results some of which are relegated to the online Supporting Information, Section 5 application to an oil price data, while conclusions are stated in Section 6. All technical proofs are in the Appendix and the online Supporting Information.

2 MAIN RESULTS

In this section, asymptotic properties of $\hat{q}_{n,\alpha}^{[k]}$ and the oracle prediction bounds are stated. Denote by $C^{(\nu,\beta)}(\mathbb{R})$ the space of functions whose ν-th derivative satisfies Hölder condition of order β where integer $\nu \geq 0$ and $\beta \in (0,1]$,

$$C^{(\nu,\beta)}(\mathbb{R}) = \left\{ \phi : \mathbb{R} \to \mathbb{R} \,\middle|\, \|\phi\|_{\nu,\beta} = \sup_{x_1,x_2 \in \mathbb{R}, x_1 \neq x_2} \frac{\left|\phi^{(\nu)}(x_1) - \phi^{(\nu)}(x_2)\right|}{|x_1 - x_2|^{\beta}} < +\infty \right\}.$$

The following assumptions are needed for the main Theorems 2.1, 2.2 and 2.3.

(A1) *The cumulative distribution function* $F \in C^{(1,\beta)}(\mathbb{R}), \beta \in (1/3, 1]$, $f(z) > 0$, $\forall z \in \mathbb{R}$, *and* $\sup_{z\in\mathbb{R}} f(z) = \|f\|_\infty < +\infty$.

(A2) *The innovations* $\{Z_t\}_{t=-\infty}^{\infty} \sim \mathrm{IID}(0, \sigma^2)$, *with* $\mathbb{E}|Z_t|^{6+3\eta} = M_\eta < +\infty$, *for some* $\eta \in (6/5, +\infty)$.

(A3) *The AR(p) process* $\{X_t\}_{t=-\infty}^{\infty}$ *is strictly stationary and causal, i.e.,* $\inf_{|z|\leq 1} \left|1 - \phi_1 z - \cdots - \phi_p z^p\right| > 0$.

(A4) *As* $n \to \infty$, $n^{-3/8} \ll h = h_n \ll n^{-\{2(1+\beta)\}^{-1}}$.

(A5) *The kernel function* $K(\cdot)$ *is a symmetric probability density, supported on* $[-1, 1]$, *and* $K \in C^{(2,1)}(\mathbb{R})$, *i.e.,* $\|K\|_{2,1} < +\infty$.

Remark 2.1. *Assumptions (A2)-(A3) are standard for AR(p) time series, see Wang et al. (2014). Note that Assumption (A1) and properties of convolution imply that* $F^{[k]} \in C^{(1,\beta)}(\mathbb{R})$, *and that the probability density function* $f^{[k]}(z) = dF^{[k]}(z)/dz$ *also satisfies* $f^{[k]}(z) > 0$, $\forall z \in \mathbb{R}$, *and* $\sup_{z\in\mathbb{R}} f^{[k]}(z) = \left\|f^{[k]}\right\|_\infty < +\infty$.

Remark 2.2. *Assumption (A3) ensures that* X_t *is an infinite moving average*

$$\begin{aligned} X_t &= \sum\nolimits_{j=0}^{\infty} \psi_j Z_{t-j}, a.s., t \in \mathbb{Z}, \\ \sum_{j=0}^{\infty} \psi_j z^j &= 1/\phi(z) = \left(1 - \phi_1 z - \cdots - \phi_p z^p\right)^{-1}, \end{aligned} \tag{2.1}$$

while equation (3.3.6) of Brockwell and Davis (1991) ensures that

$$\left|\psi_j\right| \leq C_\psi \rho_\psi^j, C_\psi > 0, 0 < \rho_\psi < 1, j \in \mathbb{N}. \tag{2.2}$$

Furthermore, the above imply that

$$\left\{\mathrm{E}|X_t|^{6+3\eta}\right\}^{1/(6+3\eta)} \leq \sum\nolimits_{j=0}^{\infty} C_\psi \rho_\psi^j \left\{\mathrm{E}|Z_{t-j}|^{6+3\eta}\right\}^{1/(6+3\eta)} < \infty. \tag{2.3}$$

Remark 2.3. *Theorem 3, p.91, Doukhan (1994) and Assumptions (A1), (A3) ensure that the induced p-dimensional Markov chain* $\{\mathbf{Y}_t = (X_t, \ldots, X_{t-p+1})\}_{t=-\infty}^{\infty}$ *is geometrically* β*-mixing, thus the AR(p) process* $\{X_t\}_{t=-\infty}^{\infty}$ *itself is geometrically* β*-mixing as well. Since* α*-mixing coefficient is no greater than the* β*-mixing coefficient* *according to Proposition 1, p.4, Doukhan (1994), there exist constants $C_\rho > 0$ and $\rho \in (0,1)$ such that the α-mixing coefficient $\alpha(k) \leq C_\rho \rho^k$ holds for all k, where*

$$\alpha(k) = \sup_{B \in \sigma\{X_t, t \leq s\}, C \in \sigma\{X_t, t \geq s+k\}} \left| P(B \cap C) - P(B)P(C) \right|, k \geq 1.$$

According to equation (5.5.4), p.183, Brockwell and Davis (1991), the $\tilde{X}_{n+k}^{[k]}$ defined in Section 1, equation (1.2) satisfies (with $\left\{\psi_j\right\}$ defined in (2.1))

$$\tilde{X}_{n+k}^{[k]} = \sum_{j=k}^{\infty} \psi_j Z_{n+k-j}, a.s.,$$

whereas the prediction error $Z_{n+k}^{[k]}$ is a linear combination of $\{Z_t\}_{t=n+1}^{n+k}$, i.e.

$$Z_{n+k}^{[k]} = \sum\nolimits_{j=0}^{k-1} \psi_j Z_{n+k-j}, \tag{2.4}$$

hence the linear predictor $\tilde{X}_{n+k}$ and the prediction noise $Z_{n+k}^{[k]}$ are independent. In particular, $\mathcal{L}\left(X_{n+k} - \tilde{X}_{n+k}^{[k]} \middle| X_n, X_{n-1}, X_{n-2}, ...\right) = \mathcal{L}\left(Z_{n+k}^{[k]}\right)$. By delta method, $\boldsymbol{\phi}^{[k]}$ is $\sqrt{n}$-consistent with $\hat{\boldsymbol{\phi}}^{[k]}$. Hence $\hat{X}_{n+k}^{[k]}$ defined in (1.4) approximates $\tilde{X}_{n+k}^{[k]}$ at the rate of $\mathcal{O}_p\left(n^{-1/2}\right)$.

Assume for the sake of discussion that the error sequence $\left\{Z_t^{[k]}\right\}_{t=k}^{n}$ were actually observed, two would-be estimators of $F^{[k]}(z)$ are the empirical cdf $F_n^{[k]}(z)$ and a KDE $\tilde{F}^{[k]}(z)$ given below

$$F_n^{[k]}(z) = n^{-1} \sum_{t=k}^{n} I\left(Z_t^{[k]} \leq z\right), z \in \mathbb{R} \tag{2.5}$$

$$\tilde{F}^{[k]}(z) = \int_{-\infty}^{z} \tilde{f}^{[k]}(z)\, du = n^{-1} \sum_{t=k}^{n} \int_{-\infty}^{z} K_h\left(u - Z_t^{[k]}\right) du, z \in \mathbb{R}. \tag{2.6}$$

Integral form of distribution estimator such as in (1.5) and (2.6) appeared in Reiss (1981) and more recently, Liu and Yang (2008), Wang et al. (2013) for iid and stationary sequences. To use residuals instead of unobserved errors in KDE is the innovation of Wang et al. (2014), yielding efficient and smooth distribution estimators that previously did not exist.

The following Theorem 2.1 establishes asymptotic equivalence of $\tilde{F}^{[k]}(z)$ and $F_n^{[k]}(z)$ at order $o_p\left(n^{-1/2}\right)$, similar to Wang et al. (2013). Although these "estimators" of $F^{[k]}(z)$ use unobservable $\left\{Z_t^{[k]}\right\}_{t=k}^{n}$ and are therefore infeasible, they nonetheless serve as a performance benchmark for any data-driven estimator of $F^{[k]}(z)$.

Theorem 2.1. *Under Assumptions (A1), (A4), (A5), the infeasible estimator* $\tilde{F}^{[k]}(z)$ *given in (2.6) is as efficient as the empirical cdf* $F_n^{[k]}(z)$ *in (2.5) over* $z \in \mathbb{R}$*, that is, as* $n \to \infty$, $\sup_{z\in\mathbb{R}} \left| \tilde{F}^{[k]}(z) - F_n^{[k]}(z) \right| = \mathcal{o}_p\left(n^{-1/2}\right)$.

The next Theorem 2.2 establishes asymptotic equivalence of $\hat{F}^{[k]}$ and $\tilde{F}^{[k]}$ up to order $\mathcal{o}_p\left(n^{-1/2}\right)$, similar to Wang et al. (2014).

Theorem 2.2. *Under Assumptions (A1)-(A5), the oracle estimator* $\hat{F}^{[k]}(z)$ *given in (1.5) is asymptotically as efficient as the infeasible estimator* $\tilde{F}^{[k]}(z)$ *over* $z \in \mathbb{R}$*, that is, as* $n \to \infty$, $\sup_{z\in\mathbb{R}} \left| \hat{F}^{[k]}(z) - \tilde{F}^{[k]}(z) \right| = \mathcal{o}_p\left(n^{-1/2}\right)$.

For any $\alpha \in (0,1)$, the next Theorem 2.3 provides the asymptotic $100(1-\alpha)\%$ oracle PI for X_{n+k}, whose implementation is described in Section 3, with data-driven predictor $\hat{X}_{n+k}^{[k]}$ is defined in (1.4).

Theorem 2.3. *Under Assumptions (A1)-(A5), for any* $\alpha \in (0,1)$*, as* $n \to \infty$,

$$\mathrm{P}\left(X_{n+k} \in \left[\hat{X}_{n+k}^{[k]} + \hat{q}_{n,\alpha/2}^{[k]}, \hat{X}_{n+k}^{[k]} + \hat{q}_{n,1-\alpha/2}^{[k]}\right]\right) \to 1-\alpha.$$

Proofs of these Theorems are in the Appendix.

3 IMPLEMENTATION

Based on a length $n+p$ realization $\{X_t\}_{t=1-p}^{n}$ of AR(p) time series, one obtains first the Yule-Walker estimator $\hat{\boldsymbol{\phi}}$ of $\boldsymbol{\phi} = \left(\phi_1, ..., \phi_p\right)^T$

$$\begin{aligned}
\hat{\boldsymbol{\phi}} &= \hat{\boldsymbol{\Gamma}}_p^{-1}\hat{\boldsymbol{\gamma}}_p, \hat{\boldsymbol{\Gamma}}_p = \{\hat{\gamma}(i-j)\}_{i,j=1}^{p}, \hat{\boldsymbol{\gamma}}_p = (\hat{\gamma}(1), ..., \hat{\gamma}(p))^T, \\
\hat{\gamma}(l) &= n^{-1}\sum\nolimits_{i=1-p}^{n-|l|} X_i X_{i+l}, l = 0, \pm 1, ..., \pm p.
\end{aligned}$$

Then, the k-step ahead innovations are estimated by $\hat{Z}_t^{[k]} = X_t - \hat{X}_t^{[k]} = X_t - \hat{\phi}_1^{[k]} X_{t-k} - \cdots - \hat{\phi}_p^{[k]} X_{t-k-p+1}$, $k \le t \le n$ where $\hat{\boldsymbol{\phi}}^{[k]} = \left(\hat{\phi}_1^{[k]}, ..., \hat{\phi}_p^{[k]}\right)^{\mathsf{T}} = g_k\left(\hat{\boldsymbol{\phi}}\right)$ is the plug-in Yule-Walker estimator of $g_k(\boldsymbol{\phi})$. For example, if $p=1, k=2$, then $\hat{Z}_t^{[2]} = X_t - \hat{\phi}_1^{[2]} X_{t-2} = X_t - \hat{\phi}_1^2 X_{t-2}$, $2 \le t \le n$, where $\hat{\phi}_1$ is the Yule-Walker estimator of ϕ_1.

The quantile estimator $\hat{q}_{n,\alpha}^{[k]} = \left(\hat{F}^{[k]}\right)^{-1}(\alpha)$ is the value z from the 1001 equally-spaced grid points from $\min\left(\left\{\hat{Z}_t^{[k]}\right\}_{t=k}^{n}\right) - h$ to $\max\left(\left\{\hat{Z}_t^{[k]}\right\}_{t=k}^{n}\right) + h$ with step size $\left\{\max\left(\left\{\hat{Z}_t^{[k]}\right\}_{t=k}^{n}\right) - \min\left(\left\{\hat{Z}_t^{[k]}\right\}_{t=k}^{n}\right) + 2h\right\}/1000$, such that $\hat{F}^{[k]}(z)$ defined by

(1.5) is the closest to α. The estimated cdf $\hat{F}^{[k]}(z)$ is computed with triweight kernel $K(u) = 35(1-u^2)^3 \times I\{|u| \leq 1\}/32$ which satisfies Assumption (A5), and a data-driven bandwidth $h = \text{IQR} \times n^{-1/3}$, where IQR denotes the Sample Inter-Quartile Range of $\left\{\hat{Z}_t^{[k]}\right\}_{t=k}^{n}$. As pointed out in Wang et al. (2014), this robust bandwidth satisfies Assumption (A4) as long as the Hölder order $\beta > 1/2$.

4 SIMULATION

4.1 Simulation Setup

In the following simulation study, three combinations of AR models and candidate error distributions $F(z)$ are involved:

Case 1 (Normal). AR(1) with standard normal distribution $N(0,1)$;

Case 2 (Kurtotic). AR(2) with kurtotic unimodal normal mixture distribution $\frac{2}{3}N(0,1) + \frac{1}{3}N\left(0, (\frac{1}{10})^2\right)$;

Case 3 (Bimodal). AR(2) with separated bimodal normal mixture distribution $\frac{1}{2}N\left(-\frac{3}{2}, (\frac{1}{2})^2\right) + \frac{1}{2}N\left(\frac{3}{2}, (\frac{1}{2})^2\right)$.

Obviously, the above three candidate distributions satisfy Assumptions (A1)-(A2) on $F(z)$, and represent rather diverse density shapes as well, see Marron and Wand (1992), which introduced more details on the richness of normal mixture distributions. A number of other combinations of AR models and error distributions have been examined with similar results, and have been reported omitted.

For all examples, the sample size n is taken to be $50, 100, 500, 1000$ and realizations $\{X_t\}_{t=-999}^{n}$ of size $1000+n$ generated, with the first 1000 values thrown out to ensure strict stationarity of $\{X_t\}_{t=1}^{n}$.

4.2 Prediction Bounds for $X_{n+k}, k = 2, 3$

In the following, we construct 95% infeasible, oracle, normal, empirical and bootstrap PIs mentioned in Section 1 and compare their performance over 1000 replications.

The infeasible PI: $\left[\tilde{X}_{n+k}^{[k]} + q_{0.025}^{[k]}, \tilde{X}_{n+k}^{[k]} + q_{0.975}^{[k]}\right]$ given in (1.3), based on presumed knowledge of the true error distribution $F^{[k]}(z)$;

The oracle PI: $\left[\hat{X}_{n+k}^{[k]} + \hat{q}_{n,0.025}^{[k]}, \hat{X}_{n+k}^{[k]} + \hat{q}_{n,0.975}^{[k]}\right]$ given in (1.6), by substituting $\hat{F}^{[k]}(z)$ for $F^{[k]}(z)$;

The normal PI: $\left[\hat{X}_{n+k}^{[k]} - 1.96\hat{\sigma}\left(k\right), \hat{X}_{n+k}^{[k]} + 1.96\hat{\sigma}\left(k\right)\right]$, where $\hat{\sigma}\left(k\right)$ is standard deviation of $\{\hat{Z}_t^{[k]}\}_{t=k}^n$, based on the naive presumption that $F^{[k]}\left(z\right)$ is normal.

The empirical PI: $\left[\hat{X}_{n+k}^{[k]} + \tilde{q}_{n,\alpha/2}^{[k]}, \hat{X}_{n+k}^{[k]} + \tilde{q}_{n,1-\alpha/2}^{[k]}\right]$ given in (1.7), based on empirical cdf of $\{\hat{Z}_t^{[k]}\}_{t=k}^n$.

The bootstrap PI: $\left[Q_B^{[k]}\left(0.025\right), Q_B^{[k]}\left(0.975\right)\right]$, where $Q_B^{[k]}\left(\alpha\right)$ is the α-th quantile of B replicates of future values $X_{n+k}^{*1}, \ldots, X_{n+k}^{*B}$ generated by the bootstrap algorithm proposed by Thombs and Schucany (1990). The default of B is set to be 1000.

Tables 1-3 show the above PIs' coverage frequencies of X_{n+2}. The coverage frequencies of oracle and empirical PIs are nearly the same for large n. Although there is no significant difference of coverage frequencies between oracle and bootstrap PIs as n increases, the computing time of bootstrap PI are much longer than that of oracle PI, see Table 4. Comparison between oracle and normal PIs is another major objective. Table 1 is for Case 1 (Normal), where coverage frequencies of normal and oracle PIs are similar, and both become closer to that of the infeasible PI with increasing n. From Table 2 for Case 2 (Kurtotic) and Table 3 for Case 3 (Bimodal), one can see the coverage frequencies of normal PI are systematically lower than the nominal value 0.95 for Case 2 and higher for Case 3, even for large sample sizes. Coverage frequencies for oracle PI, however, are consistently close to the nominal level 0.95, and become closer to those of the infeasible PI as sample size n increases. Similar results of three cases for X_{n+3} are in Tables S.1-S.3 of the online Supporting Information. All of these observations are consistent with Theorem 2.3.

We have also created boxplots for the random ratio $\left(\hat{q}_{n,0.975}^{[k]} - \hat{q}_{n,0.025}^{[k]}\right) / \left(2 \times 1.96\hat{\sigma}_{[k]}\right)$ of the lengths of oracle and normal PIs where $k = 2$ or $k = 3$ in Figures 1–3. One observes that as sample size n increases, $\left(\hat{q}_{n,0.975}^{[k]} - \hat{q}_{n,0.025}^{[k]}\right) / \left(2 \times 1.96\hat{\sigma}_{[k]}\right) \xrightarrow{p} 1$ in Case 1 (Normal), while for Case 2 (Kurtotic) and Case 3 (Bimodal), the random ratio converges respectively to values much greater and smaller than 1 respectively. It indicates that the length of normal PI may be too narrow in Case 2 and too wide in Case 3, which partly demonstrates why the coverage frequencies of normal PI are lower or higher than nominal level. Such significant difference in lengths provides another reason to use the smart oracle PI instead of the naive normal PI, in addition to the coverage frequencies discussed above.

Some may wonder why one would not prefer a PI that is better at catching future true value, such as the normal PI in Case 3 (Bimodal) with coverage frequencies higher than nominal. The short answer is: No, not at the price of precision. As shown in Figure 3, the normal PI for Case 3 (Bimodal) is on average much wider than the oracle PI, and therefore substantially less useful in locating the whereabout of the future value. The oracle PI is adaptive regardless of the distribution $F^{[k]}(z)$, and strikes an intelligent balance between coverage probability and precision.

5 APPLICATION

In this section, we analyze monthly spot price series for crude oil $(01/1986 - 12/2016)$, which can be downloaded from http://www.eia.gov/dnav/pet/pet_pri_spt_s1_m.htm. Based on $01/1986 - 01/2006$ monthly oil price series, Cryer and Chan (2008) took the first difference of logarithms of prices and further removed of its mean to make the series look more stationary than original price series in Exhibit 5.4 (pp. 91), then suggested specifying an AR(1) model was reasonable in Exhibit 6.31 (pp. 139). Table 5 respectively shows the coverage frequencies of oracle and normal prediction intervals for the last $50, 70, 100, 131$ different points under 2-step and 3-step rolling forecast. By comparison, the coverage frequencies of oracle PIs are more closer to the theoretical confidence level 95%. The Figure 4 shows 95% oracle and normal prediction intervals under 2-step rolling forecast for 131 difference points from 02/2006 to 12/2016.

6 CONCLUSIONS

In this paper we have examined the infeasible, oracle, empirical, bootstrap and normal k-step ahead PIs for future values of AR(p) time series. The oracle PI is shown to be theoretically optimal in the sense, that it performs as well as the infeasible PI with true underlying distribution of independent innovation known, and thus better than the normal PI unless the true distribution is exactly Gaussian, a mathematical fact amply borne out in numerical studies. Although as good as the oracle PI in simulation, the empirical PI does not have theoretical justification, while the bootstrap PI of Thombs and Schucany (1990) takes much longer computing time. Considering all of the above findings, we recommend with confidence the oracle PI as a robust multi-step ahead prediction tool for AR model.

Although in this paper our focus is on AR(p) model, the basic idea is applicable to other time series models such as ARMA, SARIMA, VAR, etc. In all cases, the KDE of multi-step ahead innovation distribution based on residuals can be shown oracally efficient, and thus the corresponding quantiles are efficient estimators. Hence much future work in this direction is expected, namely, prediction intervals of various types developed for other time series models such as ARMA, PARMA, VAR, by establishing uniform oracle efficiency of KDE based on residuals, as in Wang et al. (2014).

APPENDIX

A.1 Preliminaries

In what follows, we take $\|\cdot\|_2$ and $\|\cdot\|_\infty$ as L^2-norm and supremum norm, respectively. Denote by $\mathcal{O}_p$ (or $\scriptstyle{\mathcal{O}}_p$) sequences of random variables $\mathcal{O}$ (or $\scriptstyle{\mathcal{O}}$) of certain order in probability, by $\mathcal{O}_{a.s.}$ (or $\scriptstyle{\mathcal{O}}_{a.s.}$) almost surely $\mathcal{O}$ (or $\scriptstyle{\mathcal{O}}$), and by U (or u) uniformly $\mathcal{O}$ (or $\scriptstyle{\mathcal{O}}$), etc. For any real number a, one denotes by $[a]$ its integer part.

Lemma A.1. *[Dehling, Mikosch and Sørensen (2002), Theorem 4.3]Let $\left\{\xi_j\right\}_{j=-\infty}^{\infty}$ be a strictly stationary sequence of d-dimensional random vectors defined on probability space $(\Omega, \mathcal{F}, P)$, with strong mixing coefficient $\alpha(n) \ll n^{-4-2d}$ and marginal distribution function F. For $s \in \mathbb{R}^d$ write $g_n(s) = I_{\{\xi_n \leq s\}} - F(s)$. Then the series $\Gamma(s, s') = \mathbb{E}g_1(s) g_1(s') + \sum_{n\geq 2}\{\mathbb{E}g_1(s) g_n(s') + \mathbb{E}g_n(s) g_1(s')\}$ converges absolutely for $s, s' \in \mathbb{R}^d$ and defines a covariance function. There exists a sequence $\{\zeta_n(s)\}_{n=1}^{\infty}$ of independent identically distributed Gaussian processes defined on $(\Omega, \mathcal{F}, P)$, indexed by $s \in \mathbb{R}^d$, with $\mathbb{E}\zeta_1(s) = 0, \mathbb{E}\zeta_1(s)\zeta_1(s') = \Gamma(s, s'), s, s' \in \mathbb{R}^d$, and $\lambda > 0$ that depends only on d such that with probability 1*

$$\sup_{s\in\mathbb{R}^d}\left|n^{-1}\sum\nolimits_{j=1}^{n} I_{\left\{\xi_j\leq s\right\}} - F(s) - n^{-1}\sum\nolimits_{j=1}^{n}\zeta_j(s)\right| \ll n^{-1/2}(\log n)^{-\lambda}.$$

Corollary A.1. *Under Assumption (A3), there is a sequence $\left\{\zeta_n^{[k]}(z), z \in \mathbb{R}\right\}_{n=1}^{\infty}$ of Gaussian processes such that for the $\lambda > 0$ in Lemma A.1 with $d = 1$, as $n \to \infty$,*

$$\sup_{z\in\mathbb{R}}\left|F_n^{[k]}(z) - F^{[k]}(z) - n^{-1/2}\zeta_n^{[k]}(z)\right| = \mathcal{O}_{a.s}\left(n^{-1/2}(\log n)^{-\lambda}\right),$$

$$\mathbb{E}\zeta_n^{[k]}(z) \equiv 0, \mathbb{E}\zeta_n^{[k]}(z)\zeta_n^{[k]}(z') \equiv \Gamma^{[k]}(z, z'),$$

$$\Gamma^{[k]}(z, z') \equiv \sum_{|j|\leq k}\left\{\mathbb{E}I\left(Z_t^{[k]} \leq z\right)I\left(Z_{t+j}^{[k]} \leq z'\right) - F^{[k]}(z)F^{[k]}(z')\right\}, z, z' \in \mathbb{R}.$$

Proof Assumption (A3) and the definition of $Z_j^{[k]}$ in (2.4) ensure that the $Z_j^{[k]}$'s are strictly stationary and geometrically strong mixing, hence the conclusion of Lemma A.1 applies to $\xi_j = Z_j^{[k]}, \zeta_n^{[k]}(z) = n^{-1/2}\sum_{j=1}^{n}\zeta_j(z)$. Furthermore, since the sequence $\left\{Z_j^{[k]}\right\}_{j=-\infty}^{\infty}$ is k-dependent, thus the sum in $\mathbb{E}\zeta_n^{[k]}(z)\zeta_n^{[k]}(z')$ is finite. The corollary is proved. □

Lemma A.2. *[Bosq (1998), Theorem 1.4]Let $\{\xi_i\}_{i=-\infty}^{\infty}$ be a mean zero real valued process with strong mixing coefficients $\alpha(n)$. Suppose that there exists a $c > 0$ such that for $i = 1, \ldots, n, \mu \geq 3$, $\mathbb{E}|\xi_i|^\mu \leq c^{\mu-2}\mu!\mathbb{E}\xi_i^2 < +\infty$, and denote $m_r = \max_{1\leq i\leq n}\|\xi_i\|_r, r \geq 2$. Then for each $n > 1$, integer $q \in [1, n/2]$, each $\varepsilon_n > 0$ and $\mu \geq 3$*

$$\mathrm{P}\left\{\left|\sum_{i=1}^{n}\xi_i\right| > n\varepsilon_n\right\} \leq a_1 \exp\left(-\frac{q\varepsilon_n^2}{25m_2^2 + 5c\varepsilon_n}\right) + a_2(\mu)\,\alpha\left(\left[\frac{n}{q+1}\right]\right)^{2\mu/(2\mu+1)},$$

where

$$a_1 = 2\frac{n}{q} + 2\left(1 + \frac{\varepsilon_n^2}{25m_2^2 + 5c\varepsilon_n}\right), a_2(\mu) = 11n\left(1 + \frac{5m_\mu^{2\mu/(2\mu+1)}}{\varepsilon_n}\right).$$

Lemma A.3. *Under Assumptions (A2), (A3), there exists a $\gamma > 0$ such that as $n \to \infty, \max(|X_1|, |X_2|, \cdots, |X_n|) = \mathcal{O}_{a.s.}(n^\gamma)$.*

Proof By (2.1) and (2.2), $|X_t| \leq C_\psi \sum_{j=0}^{\infty}\rho_\psi^j |Z_{t-j}|, t \in \mathbb{Z}$. Define a nonnegative random variable $W = C_\psi \sum_{j=0}^{\infty}\rho_\psi^j |Z_{1-j}|$, and Assumption (A2) entails that

$$\mathbb{E}W = C_\psi \sum\nolimits_{j=0}^{\infty}\rho_\psi^j \mathbb{E}|Z_{1-j}| \leq C_\psi\left(1-\rho_\psi\right)^{-1} M_\eta^{1/(6+3\eta)} < +\infty,$$

thus $W < +\infty$ almost surely. Note that

$$\max\{|X_1|, \ldots, |X_n|\} \leq \max\{|Z_2|, \ldots, |Z_n|\}\, C_\psi\left(1-\rho_\psi\right)^{-1} + W, \tag{A.1}$$

because $|X_1| \leq W$ and

$$\begin{aligned} |X_2| &\leq C_\psi |Z_2| + \rho_\psi W, \\ &\cdots \\ |X_n| &\leq C_\psi\left(|Z_n| + \rho_\psi |Z_{n-1}| + \cdots + \rho_\psi^{n-2}|Z_2|\right) + \rho_\psi^{n-1}W. \end{aligned}$$

Under Assumption (A2),

$$\begin{aligned} \mathrm{P}\left\{\max_{2\leq t\leq n}|Z_t| > n^\gamma\right\} &\leq \sum_{t=2}^{n}\mathrm{P}\{|Z_t| > n^\gamma\} \\ &\leq \sum_{t=2}^{n} n^{-\gamma(6+3\eta)}\mathbb{E}|Z_t|^{6+3\eta} \leq n^{-\gamma(6+3\eta)+1}M_\eta, \end{aligned}$$

hence for $\gamma > 2(6+3\eta)^{-1}$

$$\sum_{n=1}^{\infty}\mathrm{P}\left\{\max_{2\leq t\leq n}|Z_t| > n^\gamma\right\} \leq \sum_{n=1}^{\infty} n^{-\gamma(6+3\eta)+1}M_\eta < +\infty,$$

so $\max_{2\le t\le n}|Z_t| = \mathcal{O}_{a.s.}(n^{\gamma})$ by Borel-Cantelli Lemma, Lemma A.3 is proved by noting (A.1). $\square$

The following Lemma A.4 follows by elementary algebra.

Lemma A.4. *Under Assumptions (A2), (A4), there exists an $a > 0$, such that the following are fulfilled for the sequence $\{D_n\} = \{n^a\}$, $\sum_{n=1}^{\infty} D_n^{-(2+\eta)} < \infty$, $D_n^{-(1+\eta)} n^{1/2} h^{1/2} \to 0$, $D_n n^{-1/2} h^{-1/2} (\log n) \to 0$.*

The next Lemma A.5 provides all the building blocks for proving Theorem 2.2. Its proof is given in the online Supporting Information.

Lemma A.5. *Under Assumptions (A1)-(A5), for any $k \le r,s,v,w \le k+p-1$, as $n \to \infty$,*

$$\sup_{z\in\mathbb{R}} \left| n^{-1} \sum_{t=k}^{n} K_h\left(z - Z_t^{[k]}\right) X_{t-r} \right| = \mathcal{O}_p\left(n^{-1/2} h^{-1/2} \log n\right), \tag{A.2}$$

$$\sup_{z\in\mathbb{R}} \left| n^{-1} \sum_{t=k}^{n} K_h'\left(z - Z_t^{[k]}\right) X_{t-r} X_{t-s} \right| = \mathcal{O}_p(1), \tag{A.3}$$

$$\sup_{z\in\mathbb{R}} \left| n^{-1} \sum_{t=k}^{n} K_h''\left(z - Z_t^{[k]}\right) X_{t-r} X_{t-s} X_{t-v} \right| = \mathcal{O}_p(1), \tag{A.4}$$

$$n^{-1} \sum_{t=k}^{n} \left| X_{t-r} X_{t-s} X_{t-v} X_{t-w} \right| = \mathcal{O}_p(1). \tag{A.5}$$

In what follows, we take $\alpha \in (0,1)$. Recall the oracle quantile estimator $\hat{q}_{n,\alpha}^{[k]} = \left(\hat{F}^{[k]}\right)^{-1}(\alpha) = \inf\left\{x : \hat{F}^{[k]}(x) \ge \alpha\right\}$, and its basic property:

Lemma A.6. *Under Assumptions (A1)-(A5), for any $\alpha \in (0,1)$, as $n \to \infty$, $\left|\hat{q}_{n,\alpha}^{[k]} - q_\alpha^{[k]}\right| = \mathcal{O}_p\left(n^{-1/2}\right)$.*

Proof It follows from Theorems 2.1, 2.2 and Corollary A.1 that

$$A_n = \sup\nolimits_{z\in\mathbb{R}} \left| \hat{F}^{[k]}(z) - F^{[k]}(z) \right| = \mathcal{O}_p\left(n^{-1/2}\right). \tag{A.6}$$

As noted in Remark 2.1, $F^{[k]}$ has everywhere positive derivative and there exists a unique $q_\alpha^{[k]}$ with $F^{[k]}\left(q_\alpha^{[k]}\right) = \alpha$. It is elementary to verify that $\hat{F}^{[k]}(z)$ is strictly increasing for $\hat{F}^{[k]}(z) \in (0,1)$, therefore, $\hat{F}^{[k]}\left(\hat{q}_{n,\alpha}^{[k]}\right) = \alpha$. According to the definition of A_n in (A.6), one has $\left|\hat{F}^{[k]}\left(\hat{q}_{n,\alpha}^{[k]}\right) - F^{[k]}\left(\hat{q}_{n,\alpha}^{[k]}\right)\right| \le A_n$, thus $\left|\alpha - F^{[k]}\left(\hat{q}_{n,\alpha}^{[k]}\right)\right| \le A_n$, so

$$\left|F^{[k]}\left(\hat{q}_{n,\alpha}^{[k]}\right) - F^{[k]}\left(q_\alpha^{[k]}\right)\right| \le A_n. \tag{A.7}$$

Since $A_n = \mathcal{O}_p\left(n^{-1/2}\right)$, $\forall\varepsilon > 0$, $\exists M > 0$, s.t. $\mathrm{P}\left(\omega\left||A_n(\omega)| > n^{-1/2}M\right.\right) < \varepsilon/2$, $n = 1, 2, ...,$ letting $\Omega_n = \left\{\omega\left|\left|F^{[k]}\left(\hat{q}^{[k]}_{n,\alpha}(\omega)\right) - F^{[k]}\left(q^{[k]}_\alpha\right)\right| > n^{-1/2}M\right.\right\}$, then (A.7) implies that $\mathrm{P}(\Omega_n) < \varepsilon/2, \forall n = 1, 2, ...$ Monotonicity of $F^{[k]}(z)$ (see Remark 2.1) entail that $\exists\delta > 0$, such that

$$\left|F^{[k]}(z) - F^{[k]}\left(q^{[k]}_\alpha\right)\right| \geq \begin{cases} 1/2 f^{[k]}\left(q^{[k]}_\alpha\right)\left|z - q^{[k]}_\alpha\right|, \text{ if } \left|z - q^{[k]}_\alpha\right| \leq \delta, \\ C_\delta, \text{ for some constant } C_\delta > 0, \text{ if } \left|z - q^{[k]}_\alpha\right| > \delta. \end{cases}$$

Let $\Delta_n = \left\{\omega\left|\left|\hat{q}^{[k]}_{n,\alpha}(\omega) - q^{[k]}_\alpha\right| \leq \delta\right.\right\}$, then $\overline{\Delta}_n = \left\{\omega\left|\left|\hat{q}^{[k]}_{n,\alpha}(\omega) - q^{[k]}_\alpha\right| > \delta\right.\right\}$. For $\omega \in \Delta_n$, $\left|F^{[k]}\left(\hat{q}^{[k]}_{n,\alpha}(\omega)\right) - F^{[k]}\left(q^{[k]}_\alpha\right)\right| \geq 1/2 f^{[k]}\left(q^{[k]}_\alpha\right)\left|\hat{q}^{[k]}_{n,\alpha}(\omega) - q^{[k]}_\alpha\right|$, i.e.,

$$\left|\hat{q}^{[k]}_{n,\alpha} - q^{[k]}_\alpha\right| \leq 2\left\{f^{[k]}\left(q^{[k]}_\alpha\right)\right\}^{-1}\left|F^{[k]}\left(\hat{q}^{[k]}_{n,\alpha}(\omega)\right) - F^{[k]}\left(q^{[k]}_\alpha\right)\right|.$$

For $\omega \in \overline{\Delta}_n$, $\left|F^{[k]}\left(\hat{q}^{[k]}_{n,\alpha}(\omega)\right) - F^{[k]}\left(q^{[k]}_\alpha\right)\right| \geq C_\delta > 0$. If $n > M/C_\delta^2$, then $n^{-1/2}M < C_\delta$, so $\omega \in \Omega_n$, hence $\mathrm{P}\left(\overline{\Delta}_n\right) \leq \mathrm{P}(\Omega_n) < \varepsilon/2$, for $n > n_0 = \left[(M/C_\delta)^2\right]$, and therefore

$$\mathrm{P}\left(\Delta_n \cap \overline{\Omega}_n\right) = 1 - \mathrm{P}\left(\overline{\Delta}_n \cup \Omega_n\right) \geq 1 - \varepsilon,$$

while $\omega \in \Delta_n \cap \overline{\Omega}_n$ implies that

$$\left|\hat{q}^{[k]}_{n,\alpha}(\omega) - q^{[k]}_\alpha\right| \leq 2\left\{f^{[k]}\left(q^{[k]}_\alpha\right)\right\}^{-1}\left|F^{[k]}\left(\hat{q}^{[k]}_{n,\alpha}(\omega)\right) - F^{[k]}\left(q^{[k]}_\alpha\right)\right|$$

and $\left|F^{[k]}\left(\hat{q}^{[k]}_{n,\alpha}(\omega)\right) - F^{[k]}\left(q^{[k]}_\alpha\right)\right| \leq n^{-1/2}M$, and thus

$$\left|\hat{q}^{[k]}_{n,\alpha}(\omega) - q^{[k]}_\alpha\right| \leq 2\left\{f^{[k]}\left(q^{[k]}_\alpha\right)\right\}^{-1} n^{-1/2}M.$$

Consequently,

$$\mathrm{P}\left(\sqrt{n}\left|\hat{q}^{[k]}_{n,\alpha} - q^{[k]}_\alpha\right| \leq 2M\left\{f^{[k]}\left(q^{[k]}_\alpha\right)\right\}^{-1}\right) \geq \mathrm{P}(\Delta_n \cap \overline{\Omega}_n) \geq 1 - \varepsilon,$$

i.e.,

$$\mathrm{P}\left(\sqrt{n}\left|\hat{q}^{[k]}_{n,\alpha} - q^{[k]}_\alpha\right| > 2M\left\{f^{[k]}\left(q^{[k]}_\alpha\right)\right\}^{-1}\right) \leq \varepsilon, \ n \geq n_0 + 1,$$

which leads to conclusion that $\left|\hat{q}^{[k]}_{n,\alpha} - q^{[k]}_\alpha\right| = \mathcal{O}_p\left(n^{-1/2}\right)$. $\square$

A.2 Proofs of Theorems 2.1, 2.2 and 2.3

Proof of Theorem 2.1 One notes that

$$\tilde{F}^{[k]}(z) = n^{-1}\sum_{t=k}^{n}\int_{-\infty}^{z} K_h\left(u - Z^{[k]}_t\right)du = \int_{-1}^{1} F^{[k]}_n(z - hv)\,K(v)\,dv,$$

hence $\tilde{F}^{[k]}(z) - F_n^{[k]}(z) = \int_{-1}^{1} \left\{ F_n^{[k]}(z - hv) - F_n^{[k]}(z) \right\} K(v)\, dv$. Corollary A.1 implies that, as $n \to \infty$,

$$\sup_{z \in \mathbb{R}} \left| F_n^{[k]}(z) - F^{[k]}(z) - n^{-1/2} \zeta_n^{[k]}(z) \right| = \mathcal{O}_{a.s.}\left(n^{-1/2} (\log n)^{-\lambda} \right).$$

The equicontinuity of $\zeta_n^{[k]}(z), z \in \mathbb{R}$ implies the following

$$\sup_{v \in [-1,1]} \sup_{z \in \mathbb{R}} \left| \sqrt{n} \left\{ F_n^{[k]}(z - hv) - F_n^{[k]}(z) \right\} - \sqrt{n} \left\{ F^{[k]}(z - hv) - F^{[k]}(z) \right\} \right| = o_p(1).$$

Thus, one obtains

$$\begin{aligned} & \int_{-1}^{1} \left\{ F_n^{[k]}(z - hv) - F_n^{[k]}(z) \right\} K(v)\, dv - \int_{-1}^{1} \left\{ F^{[k]}(z - hv) - F^{[k]}(z) \right\} K(v)\, dv \\ = & \; u_p\left(n^{-1/2} \right). \end{aligned}$$

Under Assumption (A1), applying Taylor expansion to $F^{[k]} \in C^{(1,\beta)}(\mathbb{R})$, there exists $c > 0$ such that for $z \in \mathbb{R}, v \in [-1,1]$,

$$\left| F^{[k]}(z - hv) - F^{[k]}(z) - f^{[k]}(z)(-hv) \right| \le c h^{1+\beta} |v|^{1+\beta} \le c h^{1+\beta},$$

which entails that

$$\begin{aligned} & \left| \int_{-1}^{1} \left\{ F^{[k]}(z - hv) - F^{[k]}(z) - f^{[k]}(z)(-hv) \right\} K(v)\, dv \right| \\ \le & \; c h^{1+\beta} \int_{-1}^{1} K(v)\, dv \le c h^{1+\beta} = o\left(n^{-1/2} \right) \end{aligned}$$

by the fact that $h^{1+\beta} = o\left(n^{-1/2} \right)$ according to Assumption (A4). Since $\int_{-1}^{1} vK(v)\, dv = 0$, one obtains that

$$\left| \int_{-1}^{1} \left\{ F^{[k]}(z - hv) - F^{[k]}(z) \right\} K(v)\, dv \right| = o\left(n^{-1/2} \right),$$

hence,

$$\begin{aligned} & \left| \int_{-1}^{1} \left\{ F_n^{[k]}(z - hv) - F_n^{[k]}(z) \right\} K(v)\, dv \right| \\ \le & \; \left| \int_{-1}^{1} \left\{ F^{[k]}(z - hv) - F^{[k]}(z) \right\} K(v) \right| dv + u_p\left(n^{-1/2} \right) = u_p\left(n^{-1/2} \right). \end{aligned}$$

Thus, it follows that

$$\begin{aligned} & \sup_{z \in \mathbb{R}} \left| \tilde{F}^{[k]}(z) - F_n^{[k]}(z) \right| \\ = & \; \sup_{z \in \mathbb{R}} \left| \int_{-1}^{1} \left\{ F_n^{[k]}(z - hv) - F_n^{[k]}(z) \right\} K(v)\, dv \right| = o_p\left(n^{-1/2} \right). \;\square \end{aligned}$$

Proof of Theorem 2.2 According to the definitions of $\hat{F}^{[k]}(z)$ and $\tilde{F}^{[k]}(z)$ given in (1.5) and (2.6), one has

$$\hat{F}^{[k]}(z) - \tilde{F}^{[k]}(z) = n^{-1} \sum_{t=k}^{n} \left\{ G\left(\frac{z - \hat{Z}_t^{[k]}}{h}\right) - G\left(\frac{z - Z_t^{[k]}}{h}\right) \right\}, \tag{A.8}$$

where $G(z) = \int_{-\infty}^{z} K(u)\, du$. The right-hand side of equation (A.8) is by 3-rd order Taylor expansion $I_1 + I_2 + I_3 + I_4$, where

$$\begin{aligned}
I_1 &= I_1(z) = n^{-1} \sum_{t=k}^{n} K\left(\frac{z - Z_t^{[k]}}{h}\right) \frac{\hat{Z}_t^{[k]} - Z_t^{[k]}}{h}, \\
I_2 &= I_2(z) = n^{-1} \sum_{t=k}^{n} K'\left(\frac{z - Z_t^{[k]}}{h}\right) \left(\frac{\hat{Z}_t^{[k]} - Z_t^{[k]}}{h}\right)^2, \\
I_3 &= I_3(z) = n^{-1} \sum_{t=k}^{n} K''\left(\frac{z - Z_t^{[k]}}{h}\right) \left(\frac{\hat{Z}_t^{[k]} - Z_t^{[k]}}{h}\right)^3, \\
I_4 &= I_4(z) = n^{-1} \sum_{t=k}^{n} R_t = n^{-1} \sum_{t=k}^{n} R_t(z),
\end{aligned}$$

in which $|R_t| \leq \frac{\|K\|_{2,1}}{6} \left|\frac{\hat{Z}_t^{[k]} - Z_t^{[k]}}{h}\right|^4$. Straight algebra and (1.4), (1.2) provide that

$$\hat{Z}_t^{[k]} - Z_t^{[k]} = \tilde{X}_t^{[k]} - \hat{X}_t^{[k]} = \left(\phi_1^{[k]} - \hat{\phi}_1^{[k]}\right) X_{t-k} + \cdots + \left(\phi_p^{[k]} - \hat{\phi}_p^{[k]}\right) X_{t-k-p+1}.$$

According to Lemma A.5, one obtains next that

$$\begin{aligned}
\sup_{z\in\mathbb{R}} |I_1| &= \sup_{z\in\mathbb{R}} \left| n^{-1} \sum_{t=k}^{n} K\left(\frac{z - \hat{Z}_t^{[k]}}{h}\right) \frac{\hat{Z}_t^{[k]} - Z_t^{[k]}}{h} \right| \qquad \text{(A.9)} \\
&= \sup_{z\in\mathbb{R}} \left| n^{-1} \sum_{t=k}^{n} K_h\left(z - \hat{Z}_t^{[k]}\right) \sum_{j=1}^{p} \left(\phi_j^{[k]} - \hat{\phi}_j^{[k]}\right) X_{t-k-j+1} \right| \\
&\leq p \max_{1\leq j\leq p} \left|\phi_j^{[k]} - \hat{\phi}_j^{[k]}\right| \sup_{k\leq r\leq k+p-1} \sup_{z\in\mathbb{R}} \left| n^{-1} \sum_{t=k}^{n} K_h\left(z - \hat{Z}_t^{[k]}\right) X_{t-r} \right| \\
&= \mathcal{O}_p\left(n^{-1/2}\right) \mathcal{O}_{a.s.}\left(n^{-1/2} h^{-1/2} \log n\right) = o_p\left(n^{-1/2}\right),
\end{aligned}$$

similarly,

$$\sup_{z\in\mathbb{R}} |I_2| = \sup_{z\in\mathbb{R}} n^{-1} \left| \sum_{t=k}^{n} K'\left(\frac{z - Z_t^{[k]}}{h}\right) \left(\frac{\hat{Z}_t^{[k]} - Z_t^{[k]}}{h}\right)^2 \right| = o_p\left(n^{-1/2}\right) \tag{A.10}$$

$$\sup_{z\in\mathbb{R}}|I_3| = \sup_{z\in\mathbb{R}} n^{-1}\left|\sum_{t=k}^{n} K''\left(\frac{z-Z_t^{[k]}}{h}\right)\left(\frac{\hat{Z}_t^{[k]}-Z_t^{[k]}}{h}\right)^3\right| = o_p\left(n^{-1/2}\right) \tag{A.11}$$

and together with Assumptions (A4) and (A5),

$$\begin{aligned}\sup_{z\in\mathbb{R}}|I_4| &\leq C\sup_{k\leq t\leq n}\left|\frac{\hat{Z}_t^{[k]}-Z_t^{[k]}}{h}\right|^4 \leq h^{-4}p^4\left(\max_{1\leq j\leq p}\left|\phi_j^{[k]}-\hat{\phi}_j^{[k]}\right|\right)^4 \\ &\times \sup_{k\leq r,s,v,w\leq k+p-1} n^{-1}\sum_{t=k}^{n}|X_{t-r}X_{t-s}X_{t-v}X_{t-w}| \\ &= \mathcal{O}_p\left(n^{-2}h^{-4}\right)\mathcal{O}_p(1) = o_p\left(n^{-1/2}\right).\end{aligned} \tag{A.12}$$

Since $\sup_{z\in\mathbb{R}}\left|\hat{F}^{[k]}(z)-\tilde{F}^{[k]}(z)\right| \leq \sup_{z\in\mathbb{R}}(|I_1|+|I_2|+|I_3|+|I_4|)$, Theorem 2.2 is easy to be proved by (A.9), (A.10), (A.11) and (A.12). $\square$

Proof of Theorem 2.3 Let $Z^{[k]}$ be a random variable with distribution function $F^{[k]}(z)$. Recall (1.2) and that $X_{n+k} = \phi_1^{[k]}X_n + \cdots + \phi_p^{[k]}X_{n-p+1} + Z_{n+k}^{[k]}$, thus as $n\to\infty$, $X_{n+k}-\tilde{X}_{n+k}^{[k]} = Z_{n+k}^{[k]} \xrightarrow{\mathcal{L}} Z^{[k]}$. Delta method and (1.2), (1.4) imply that $\tilde{X}_{n+k}^{[k]}-\hat{X}_{n+k}^{[k]} \xrightarrow{p} 0$, therefore $X_{n+k}-\tilde{X}_{n+k}^{[k]}+\tilde{X}_{n+k}^{[k]}-\hat{X}_{n+k}^{[k]} = X_{n+k}-\hat{X}_{n+k}^{[k]} \xrightarrow{\mathcal{L}} Z^{[k]}$ by applying Slutsky's Theorem. Next, Lemma A.6 provides that $\left|\hat{q}_{n,\alpha/2}^{[k]}-q_{\alpha/2}^{[k]}\right| \xrightarrow{p} 0$, applying Slutsky's Theorem again, one obtains that

$$X_{n+k}-\hat{X}_{n+k}^{[k]}-\left(\hat{q}_{n,\alpha/2}^{[k]}-q_{\alpha/2}^{[k]}\right) \xrightarrow{\mathcal{L}} Z^{[k]}.$$

Since the distribution function $F^{[k]}$ is continuous by Remark 2.1, one has

$$\lim_{n\to\infty}\mathrm{P}\left[X_{n+k}-\hat{X}_{n+k}^{[k]}-\left(\hat{q}_{n,\alpha/2}^{[k]}-q_{\alpha/2}^{[k]}\right)\leq q_{\alpha/2}^{[k]}\right] = \mathrm{P}\left[Z^{[k]}\leq q_{\alpha/2}^{[k]}\right] = \alpha/2,$$

i.e.,

$$\mathrm{P}\left[X_{n+k}-\hat{X}_{n+k}^{[k]}\leq \hat{q}_{n,\alpha/2}^{[k]}\right] \to \alpha/2.$$

Likewise,

$$\mathrm{P}\left[X_{n+k}-\hat{X}_{n+k}^{[k]}\leq \hat{q}_{n,(1-\alpha/2)}^{[k]}\right] \to 1-\alpha/2,$$

and consequently,

$$\mathrm{P}\left[X_{n+k}-\hat{X}_{n+k}^{[k]}\leq \hat{q}_{n,(1-\alpha/2)}^{[k]}, X_{n+k}-\hat{X}_{n+k}^{[k]} > \hat{q}_{n,\alpha/2}^{[k]}\right] \to 1-\alpha/2-\alpha/2 = 1-\alpha,$$

or

$$\mathrm{P}\left(X_{n+k}\in\left[\hat{X}_{n+k}^{[k]}+\hat{q}_{n,\alpha/2}^{[k]}, \hat{X}_{n+k}^{[k]}+\hat{q}_{n,(1-\alpha/2)}^{[k]}\right]\right) \to 1-\alpha.$$

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SUPPORTING INFORMATION

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Table 1: The 95% PIs' coverage frequencies of X_{n+2} for AR(1) in Case 1 (Normal) over 1000 replications.

ϕ	n	infeasible	oracle	normal	empirical	bootstrap
-0.8	50	0.952	0.934	0.936	0.916	0.936
	100	0.938	0.931	0.933	0.931	0.938
	500	0.952	0.950	0.952	0.951	0.967
	1000	0.957	0.954	0.955	0.954	0.943
-0.4	50	0.947	0.933	0.938	0.915	0.945
	100	0.943	0.941	0.938	0.937	0.948
	500	0.952	0.952	0.951	0.952	0.963
	1000	0.957	0.957	0.959	0.958	0.943
-0.2	50	0.952	0.926	0.940	0.918	0.945
	100	0.945	0.943	0.940	0.938	0.945
	500	0.951	0.957	0.949	0.954	0.954
	1000	0.954	0.952	0.950	0.954	0.946
0.2	50	0.944	0.928	0.941	0.918	0.939
	100	0.943	0.927	0.937	0.930	0.936
	500	0.962	0.960	0.963	0.960	0.944
	1000	0.952	0.947	0.950	0.949	0.948
0.4	50	0.947	0.920	0.941	0.913	0.935
	100	0.942	0.929	0.938	0.927	0.938
	500	0.965	0.962	0.961	0.961	0.941
	1000	0.957	0.955	0.952	0.954	0.955
0.8	50	0.954	0.924	0.928	0.913	0.912
	100	0.942	0.929	0.937	0.927	0.925
	500	0.958	0.952	0.953	0.948	0.940
	1000	0.965	0.963	0.963	0.964	0.954

Table 2: The 95% PIs' coverage frequencies of X_{n+2} for AR(2) in Case 2 (Kurtotic) over 1000 replications.

(ϕ_1, ϕ_2)	n	infeasible	oracle	normal	empirical	bootstrap
$(-0.8, -0.4)$	50	0.943	0.909	0.913	0.900	0.928
	100	0.958	0.943	0.939	0.942	0.949
	500	0.950	0.947	0.938	0.948	0.935
	1000	0.951	0.948	0.942	0.949	0.944
$(0.8, -0.4)$	50	0.944	0.900	0.904	0.894	0.925
	100	0.956	0.932	0.933	0.936	0.946
	500	0.962	0.958	0.950	0.957	0.940
	1000	0.949	0.944	0.933	0.947	0.965
$(0.2, -0.1)$	50	0.939	0.911	0.905	0.906	0.943
	100	0.950	0.941	0.934	0.942	0.946
	500	0.965	0.965	0.946	0.967	0.948
	1000	0.949	0.946	0.925	0.946	0.953
$(-0.2, 0.1)$	50	0.943	0.922	0.912	0.914	0.938
	100	0.949	0.947	0.933	0.947	0.941
	500	0.961	0.957	0.942	0.957	0.950
	1000	0.949	0.949	0.930	0.949	0.951
$(0.1, -0.05)$	50	0.943	0.916	0.905	0.907	0.943
	100	0.950	0.943	0.929	0.942	0.945
	500	0.964	0.965	0.945	0.965	0.954
	1000	0.947	0.950	0.922	0.950	0.956
$(-0.1, 0.05)$	50	0.943	0.923	0.909	0.919	0.937
	100	0.951	0.949	0.933	0.949	0.938
	500	0.963	0.958	0.943	0.958	0.952
	1000	0.948	0.943	0.927	0.945	0.953

Table 3: The 95% PIs' coverage frequencies of X_{n+2} for AR(2) in Case 3 (Bimodal) over 1000 replications.

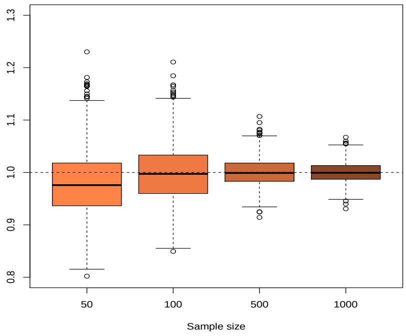
(ϕ_1, ϕ_2)	n	infeasible	oracle	normal	empirical	bootstrap
$(-0.8, -0.4)$	50	0.948	0.928	0.963	0.899	0.937
	100	0.957	0.945	0.975	0.936	0.947
	500	0.933	0.930	0.983	0.926	0.954
	1000	0.962	0.961	0.988	0.959	0.945
$(0.8, -0.4)$	50	0.951	0.912	0.947	0.883	0.930
	100	0.954	0.935	0.975	0.927	0.937
	500	0.960	0.958	0.992	0.957	0.940
	1000	0.950	0.948	0.987	0.947	0.957
$(0.2, -0.1)$	50	0.957	0.934	0.994	0.901	0.962
	100	0.956	0.951	0.992	0.925	0.953
	500	0.961	0.962	1.000	0.957	0.941
	1000	0.954	0.957	0.996	0.953	0.951
$(-0.2, 0.1)$	50	0.947	0.944	0.992	0.910	0.964
	100	0.952	0.950	0.995	0.943	0.962
	500	0.937	0.946	0.997	0.939	0.956
	1000	0.965	0.966	0.997	0.965	0.947
$(0.1, -0.05)$	50	0.957	0.939	0.995	0.900	0.970
	100	0.959	0.957	0.997	0.939	0.966
	500	0.954	0.961	1.000	0.955	0.946
	1000	0.951	0.957	0.997	0.954	0.951
$(-0.1, 0.05)$	50	0.949	0.948	0.996	0.908	0.969
	100	0.952	0.955	0.997	0.939	0.968
	500	0.942	0.952	0.997	0.947	0.957
	1000	0.962	0.963	0.998	0.962	0.950

Table 4: Computing time (minutes) of the 95% oracle and bootstrap PIs' coverage frequencies of X_{n+2} for AR(1) with coefficient -0.8 in Case 1 (Normal) over 1000 replications.

Computing time	n			
	50	100	500	1000
oracle	0.250	0.393	1.830	3.450
bootstrap	28.416	36.290	106.458	190.390
bootstrap/oracle	113.664	92.341	58.174	55.186

Table 5: Coverage frequencies of rolling forecasts by 95% oracle and normal prediction intervals for the difference series of logged oil price.

k-step ahead	number of forecasts	oracle	normal
$k = 2$	50	0.940	0.900
	70	0.957	0.929
	100	0.930	0.910
	131	0.947	0.931
$k = 3$	50	0.920	0.900
	70	0.943	0.929
	100	0.910	0.900
	131	0.931	0.924



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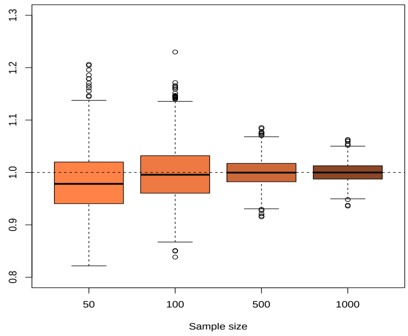

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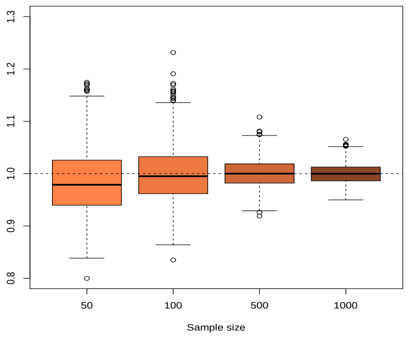

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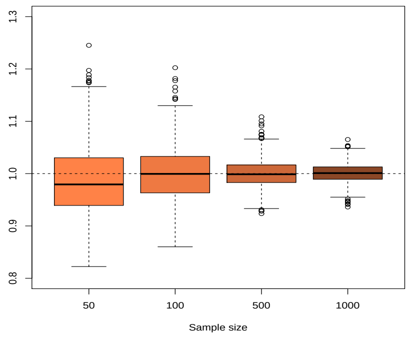

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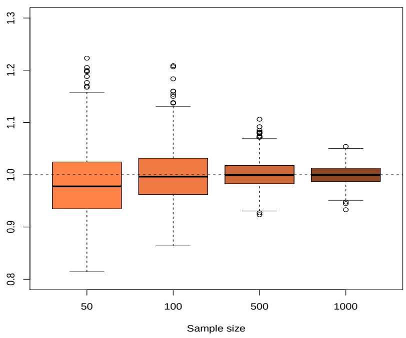

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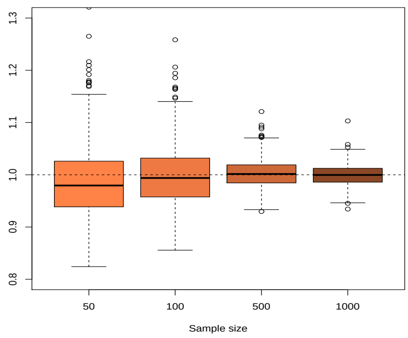
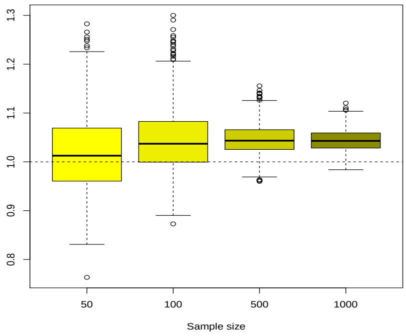
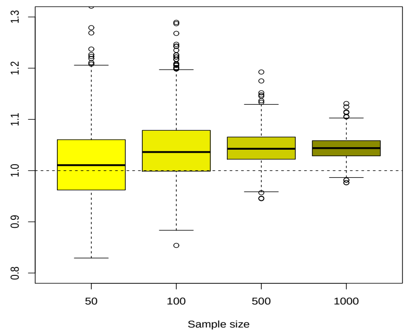


(f)

Figure 1: Boxplot of ratios $\left(\hat{q}^{[3]}_{n,0.975} - \hat{q}^{[3]}_{n,0.025}\right) / \left(2 \times 1.96\hat{\sigma}_{[3]}\right)$ for AR(1) in Case 1 (Normal). The AR coefficients of (a)-(f) are -0.8, -0.4, -0.2, 0.2, 0.4, 0.8, respectively.



(a)



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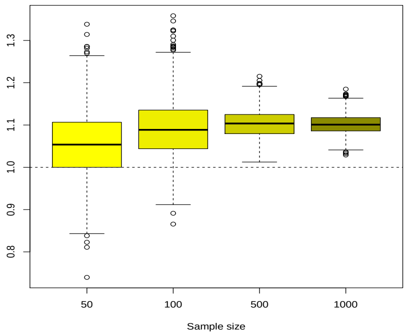

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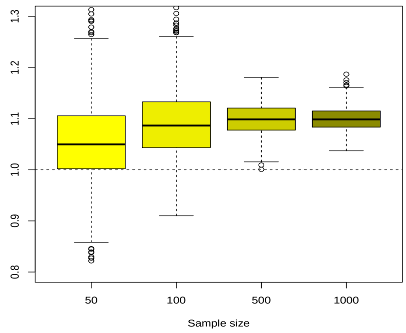

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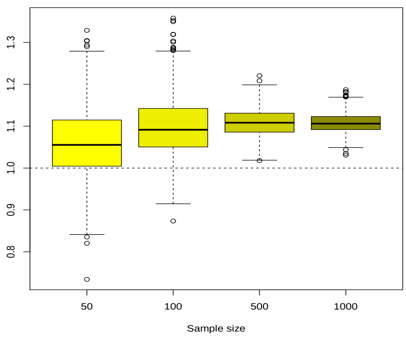

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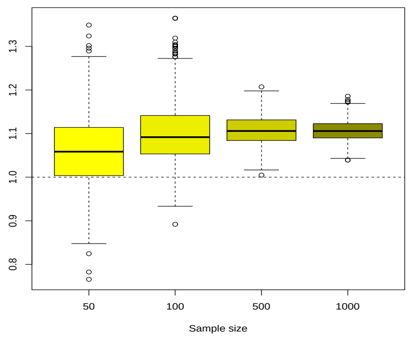
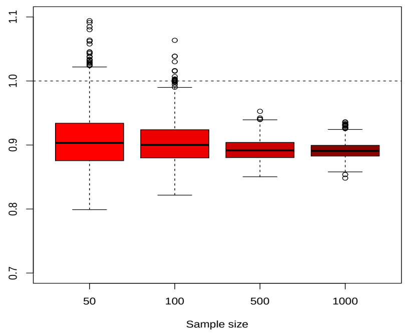
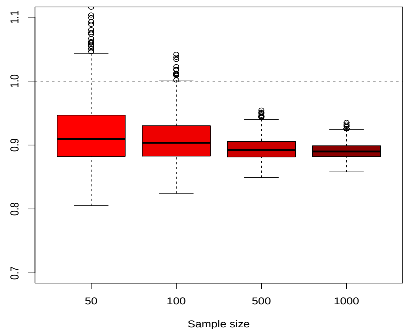


(f)

Figure 2: Boxplot of ratios $\left(\hat{q}^{[3]}_{n,0.975} - \hat{q}^{[3]}_{n,0.025}\right) / \left(2 \times 1.96\hat{\sigma}_{[3]}\right)$ for AR(2) in Case 2 (Kurtotic). The AR coefficients of (a)-(f) are $(-0.8, -0.4)$, $(0.8, -0.4)$, $(0.2, -0.1)$, $(-0.2, 0.1)$, $(0.1, -0.05)$, $(-0.1, 0.05)$ respectively.



(a)



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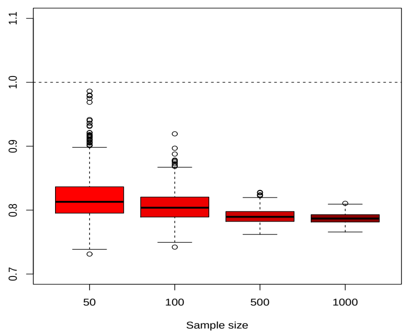

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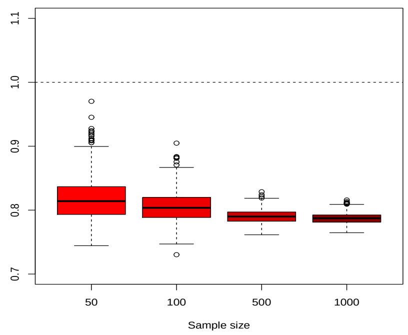

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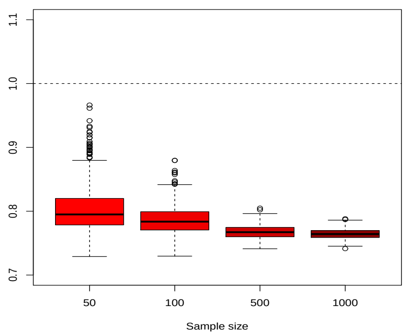

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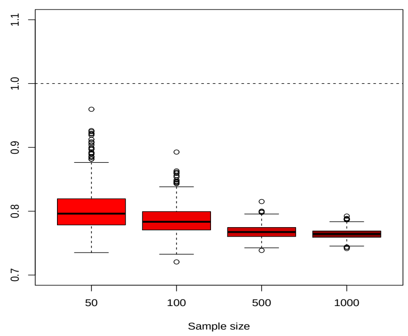

(f)

Figure 3: Boxplot of ratios $\left(\hat{q}^{[2]}_{n,0.975} - \hat{q}^{[2]}_{n,0.025}\right) / \left(2 \times 1.96\hat{\sigma}_{[2]}\right)$ for AR(2) in Case 3 (Bimodal). The AR coefficients of (a)-(f) are $(-0.8, -0.4)$, $(0.8, -0.4)$, $(0.2, -0.1)$, $(-0.2, 0.1)$, $(0.1, -0.05)$, $(-0.1, 0.05)$ respectively.

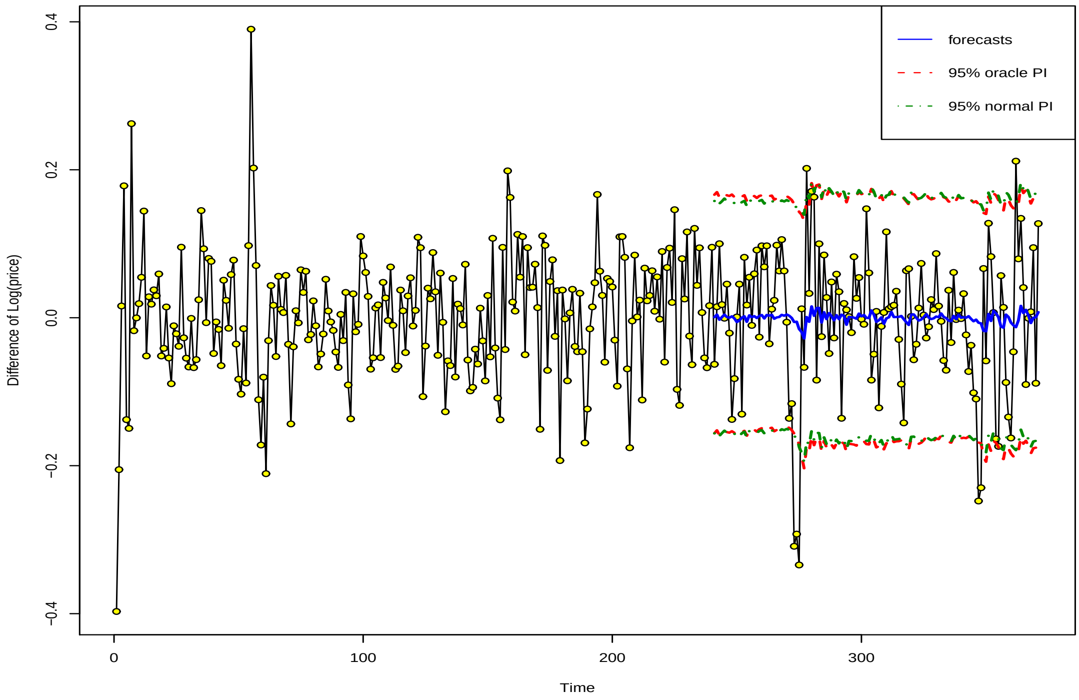

Figure 4: AR(1) model filling and 2-step ahead rolling forecasts of 131 points for the difference series of logged oil price.