STATISTICAL INFERENCE FOR FUNCTIONAL TIME SERIES: AUTOCOVARIANCE FUNCTION

Chen Zhong and Lijian Yang

Tsinghua University

Supplementary Materials

This supplement provides technical lemmas and detailed proofs of the main asymptotic results. Throughout this supplementary document, $\mathcal{O}_p$ (or $\mathcal{o}_p$) denotes a sequence of random variables of certain order in probability. For instance, $\mathcal{o}_p\left(n^{-1/2}\right)$ means a smaller order than $n^{-1/2}$ in probability, and by $\mathcal{O}_{a.s.}$ (or $\mathcal{o}_{a.s.}$) almost surely $\mathcal{O}$ (or $\mathcal{o}$). In addition, $\mathcal{U}_p$ denotes a sequence of random functions which are $\mathcal{O}_p$ uniformly in the domain.

For vector $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$, denote norms $\|\mathbf{a}\|_r = (|a_1|^r + \cdots + |a_n|^r)^{1/r}$, $r \geq 1$, $\|\mathbf{a}\|_\infty = \max(|a_1|, \ldots, |a_n|)$. For any matrix $\mathbf{A} = (a_{ij})_{i=1,j=1}^{m,n}$, denote its L_r norm as $\|\mathbf{A}\|_r = \max_{\mathbf{a}\in\mathcal{R}^n, \mathbf{a}\neq\mathbf{0}} \|\mathbf{A}\mathbf{a}\|_r \|\mathbf{a}\|_r^{-1}$, for $r < +\infty$ and $\|\mathbf{A}\|_r = \max_{1\leq i\leq m} \sum_{j=1}^n |a_{ij}|$, for $r = \infty$.

0.1 Decomposition

Let $\mathbf{B}(x) = \{B_{1,p}(x), \ldots, B_{J_s+p,p}(x)\}^\top$, the $N \times (J_s + p)$ design matrix $\mathbf{B}$ for spline regression is $\mathbf{B} = \{\mathbf{B}(1/N), \ldots, \mathbf{B}(N/N)\}^\top$ and $\mathbf{Y}_t = (Y_{t1}, \ldots, Y_{tN})^\top$, then the spline estimator $\hat{\eta}_t(x)$ in (2.6) can be represented as $\hat{\eta}_t(x) = \mathbf{B}(x)^\top \left(\mathbf{B}^\top\mathbf{B}\right)^{-1} \mathbf{B}^\top\mathbf{Y}_t$. Define the empirical inner product matrix of B-spline basis $\{B_{\ell,p}(x)\}_{\ell=1}^{J_s+p}$ as

$$\mathbf{V}_{n,p} = \left\{\langle B_{\ell,p}, B_{\ell',p}\rangle_N\right\}_{\ell,\ell'=1}^{J_s+p} = N^{-1}\mathbf{B}^\top\mathbf{B},$$

and, according to Lemma A.3 in Cao et al. (2012), for some constant $C_p > 0$, we have

$$\left\|\mathbf{V}_{n,p}^{-1}\right\|_\infty \leq C_p J_s. \tag{S.1}$$

Denote $\boldsymbol{\eta}_t = \{\eta_t(1/N), \ldots, \eta_t(N/N)\}^\top$, $\mathbf{m} = \{m(1/N), \ldots, m(N/N)\}^\top$, $\boldsymbol{\xi}_t = \{\chi_t(1/N), \ldots, \chi_t(N/N)\}^\top$, $\boldsymbol{\varepsilon}_t = \{\sigma(1/N)\varepsilon_{t1}, \ldots, \sigma(N/N)\varepsilon_{tN}\}^\top$. According to model (1.5), $\boldsymbol{\eta}_t = \mathbf{m} + \boldsymbol{\chi}_t$, then the approximation error $\hat{\eta}_t(x) - \eta_t(x)$ can be decomposed into the following:

$$\hat{\eta}_t(x) - \eta_t(x) = \tilde{\eta}_t(x) - \eta_t(x) + \tilde{\varepsilon}_t(x), \tag{S.2}$$

where

$$\tilde{\eta}_t(x) = N^{-1}\mathbf{B}(x)^\top \mathbf{V}_{n,p}^{-1}\mathbf{B}^\top\boldsymbol{\eta}_t = \tilde{m}(x) + \tilde{\chi}_t(x), \tag{S.3}$$

$$\tilde{m}(x) = N^{-1}\mathbf{B}(x)^\top \mathbf{V}_{n,p}^{-1}\mathbf{B}^\top\mathbf{m},\ \tilde{\chi}_t(x) = N^{-1}\mathbf{B}(x)^\top \mathbf{V}_{n,p}^{-1}\mathbf{B}^\top\boldsymbol{\chi}_t,$$

$$\tilde{\varepsilon}_t(x) = N^{-1}\mathbf{B}(x)^\top \mathbf{V}_{n,p}^{-1}\mathbf{B}^\top\boldsymbol{\varepsilon}_t, \tag{S.4}$$

Thus, one has $\hat{\eta}_t(x) - \eta_t(x) = \tilde{\chi}_t(x) - \chi_t(x) + \tilde{m}(x) - m(x) + \tilde{\varepsilon}_t(x)$. Therefore, by (2.5) and (S.2), the approximation error of $\hat{\chi}_t(x)$ in (2.5) to $\chi_t(x)$ can be represented by

$$\hat{\chi}_t(x) - \chi_t(x) = \tilde{\chi}_t(x) - \chi_t(x) + \tilde{\varepsilon}_t(x) - n^{-1}\sum_{t'=1}^{n}\{\tilde{\chi}_{t'}(x) + \tilde{\varepsilon}_{t'}(x)\}. \tag{S.5}$$

0.2 Technical Lemmas

In this section, we provide some technical lemmas.

Lemma S.1. *Let $W_i \sim N(0, \sigma_i^2), \sigma_i > 0$, $1 \leq i \leq n$, for $a > 2$*

$$\mathbb{P}\left(\max_{1\leq i\leq n}|W_i/\sigma_i| > a\sqrt{\log n}\right) < \sqrt{2/\pi}n^{1-a^2/2}. \tag{S.6}$$

Hence, $\left(\max_{1\le i\le n}|W_i|\right)/\left(\max_{1\le i\le n}\sigma_i\right)\le \max_{1\le i\le n}|W_i/\sigma_i| = \mathcal{O}_{a.s.}\left(\sqrt{\log n}\right)$.

Proof. Note that

$$\mathbb{P}\left(\max_{1\le i\le n}\left|\frac{W_i}{\sigma_i}\right| > a\sqrt{\log n}\right) \le \sum_{i=1}^{n}\mathbb{P}\left(\left|\frac{W_i}{\sigma_i}\right| > a\sqrt{\log n}\right) \le 2n\left\{1-\Phi\left(a\sqrt{\log n}\right)\right\}$$
$$< 2n\frac{\phi\left(a\sqrt{\log n}\right)}{a\sqrt{\log n}} \le 2n\phi\left(a\sqrt{\log n}\right) = \sqrt{2/\pi}n^{1-a^2/2},$$

for $n>2, a>1$, which proves (S.6). The lemma follows by applying Borel-Cantelli Lemma with large enough a.

Lemma S.2 (Theorem 2.6.7 of Csörgö and Révész (1981))**.** *Suppose that* ξ_i, $1\le i\le n$ *are i.i.d with* $\mathrm{E}(\xi_1)=0$, $\mathrm{E}\left(\xi_1^2\right)=1$ *and* $H(x)>0$ $(x\ge 0)$ *is an increasing continuous function such that* $x^{-2-\gamma}H(x)$ *is increasing for some* $\gamma>0$ *and* $x^{-1}\ln H(x)$ *is decreasing with* $\mathrm{E}H(|\xi_1|)<\infty$*. Then there exist constants* C_1, C_2, $a>0$ *which depend only on the distribution of* ξ_1 *and a sequence of Brownian motions* $\{W_n(m)\}_{n=1}^{\infty}$*, such that for any* $\{x_n\}_{n=1}^{\infty}$ *satisfying* $H^{-1}(n)<x_n<C_1(n\ln n)^{1/2}$ *and* $S_m=\sum_{i=1}^{m}\xi_i$*, then* $\mathbb{P}\left\{\max_{1\le m\le n}|S_m-W_n(m)|>x_n\right\}\le C_2 n\left\{H(ax_n)\right\}^{-1}$.

Lemma S.3. *Assumptions (A4) and (A5') imply Assumption (A5).*

Proof. Under Assumption (A5'), $\mathrm{E}|\zeta_{tk}|^{r_1}<+\infty$, $r_1>4+2\omega$, $\mathrm{E}|\varepsilon_{ij}|^{r_2}<+\infty$, $r_2>4+2\theta$, where ω is defined in Assumption (A4) and θ is defined in Assumption (A3), so there exists some $\beta_0,\beta_2\in(0,1/2)$, such that $r_1>(2+\omega)/\beta_0$, $r_2>(2+\theta)/\beta_2$.

Let $H(x)=x^{r_1}$. Lemma S.2 entails that there exist constants c_{1k} and a_k depending on the distribution of ζ_{tk}, such that for $x_n=(n+I_n)^{\beta_0}$, $(n+I_n)/H(a_k x_n)=a_k^{-r_1}(n+I_n)^{1-r_1\beta_0}$ and i.i.d $N(0,1)$ variables $Z_{tk,\zeta}$,

$$\mathbb{P}\left\{\max_{-I_n+1\le\tau\le n}\left|\sum_{t=-I_n+1}^{\tau}\zeta_{tk}-\sum_{t=-I_n+1}^{\tau}Z_{tk,\zeta}\right|>(n+I_n)^{\beta_0}\right\}<c_{1k}a_k^{-r_1}(n+I_n)^{1-r_1\beta_0}.$$

Since $\{\zeta_{tk}\}_{t=-I_n+1,k=1}^{n,k_n}$ are strong white noise, there is a common $c_1 > 0$, such that

$$\max_{1\le k\le k_n} \mathbb{P}\left\{ \max_{-I_n+1\le\tau\le n} \left| \sum_{t=-I_n+1}^{\tau} \zeta_{tk} - \sum_{t=-I_n+1}^{\tau} Z_{tk,\zeta} \right| > (n+I_n)^{\beta_0} \right\} < c_1 (n+I_n)^{1-r_1\beta_0}.$$

Then one can take $0 < \epsilon < 1/2 - \beta_0$ and $\beta_1 = \beta_0 + \epsilon$ such that $\beta_1 < 1/2$. Since $I_n \asymp \log n$ and $\log(1 + I_n/n)/\log n \to 0$ by (3.1), one can obtain $\epsilon > \log(1 + I_n/n)/\log n$ for large enough n, such that $(n + I_n)^{\beta_0} < n^{\beta_0+\epsilon} = n^{\beta_1}$. Consequently, there is a constant $C_1 > 0$ such that

$$\begin{aligned} \mathbb{P}\left\{ \max_{1\le k\le k_n} \max_{-I_n+1\le\tau\le n} \left| \sum_{t=-I_n+1}^{\tau} \zeta_{tk} - \sum_{t=-I_n+1}^{\tau} Z_{tk,\zeta} \right| > n^{\beta_1} \right\} &< k_n c_1 (n+I_n)^{1-r_1\beta_0} \\ &\le C_1 n^{1-r_1\beta_0+\omega}. \end{aligned}$$

By denoting $\gamma_1 = r_1\beta_0 - 1 - \omega > 1$, one obtains that

$$\mathbb{P}\left\{ \max_{1\le k\le k_n} \max_{-I_n+1\le\tau\le n} \left| \sum_{t=-I_n+1}^{\tau} \zeta_{tk} - \sum_{t=-I_n+1}^{\tau} Z_{tk,\zeta} \right| > n^{\beta_1} \right\} < C_1 n^{-\gamma_1}.$$

Likewise, under Assumption (A5'), taking $H(x) = x^{r_2}$, Lemma S.2 implies that there exists constants $c_2 > 0$ and b depending on the distribution of ε_{tj}, such that for $x_N = N^{\beta_2}$, $N/H(ax_N) = b^{-r_2} c_2^{-r_2} N^{1-r_2\beta_2}$ and iid standard normal random variables $Z_{tj,\varepsilon}$ such that

$$\max_{1\le t\le n} \mathbb{P}\left\{ \max_{1\le\tau\le N} \left| \sum_{j=1}^{\tau} \varepsilon_{tj} - \sum_{j=1}^{\tau} Z_{tj,\varepsilon} \right| > N^{\beta_2} \right\} < c_2^{-r_2} b^{-r_2} N^{1-\gamma_2\beta_2},$$

and consequently there is a $C_2 > 0$ such that

$$\mathbb{P}\left\{ \max_{1\le t\le n} \max_{1\le\tau\le N} \left| \sum_{j=1}^{\tau} \varepsilon_{tj} - \sum_{j=1}^{\tau} Z_{tj,\varepsilon} \right| > N^{\beta_2} \right\} < c_2^{-r_2} b^{-r_2} n \times N^{1-\gamma_2\beta_2} \le C_2 N^{\theta+1-\gamma_2\beta_2}.$$

Since $r_2\beta_2 > (2+\theta)$, there is $\gamma_2 = r_2\beta_2 - 1 - \theta > 1$ and Assumption (A5) follows. The lemma holds consequently.

Lemma S.4. *Under Assumption (A5), there are constants $C_3, C_4 \in (0, \infty)$ and $N(0,1)$ variables $Z_{tk,\xi} = \sum_{t'=0}^{\infty} a_{t'k} Z_{t-t',k,\zeta}, t = 1, \ldots, n, k = 1, \ldots, k_n$, $\mathrm{Cov}(Z_{jk,\xi}, Z_{j+h,k,\xi}) = \sum_{m=0}^{\infty} a_{mk} a_{m+h,k}, 1 \leq j \leq n, 0 \leq h \leq n-j$, such that for γ_1, β_1 in Assumption (A5)*

$$\mathbb{P}\left\{\max_{1\leq k\leq k_n}\max_{1\leq\tau\leq n}\left|\sum_{t=1}^{\tau}\xi_{tk} - \sum_{t=1}^{\tau} Z_{tk,\xi}\right| > C_3 n^{\beta_1}\right\} < C_4 n^{-\gamma_1}.$$

Proof. Since $\sum_{t=0}^{\infty} a_{tk}^2 = 1$ and $|a_{tk}| < C_a \rho_a^t, \forall t, k$, together with $I_n > -10\log n/\log\rho_a$ in (3.1), then $\rho_a^{I_n} < n^{-10}, |a_{t'k}| < C_a n^{-10}\rho_a^{t'-I_n}$, when $t' > I_n$ and there exits a constant M, such that $\sum_{t=0}^{I_n} |a_{tk}| < M$. It can be shown that

$$\xi_{tk} = \left(\sum_{t'=0}^{I_n} + \sum_{t'=I_n+1}^{\infty}\right) a_{t'k}\zeta_{t-t',k},$$

$$\begin{aligned}\left|\xi_{tk} - \sum_{t'=0}^{I_n} a_{t'k}\zeta_{t-t',k}\right| &\leq \sum_{t'=I_n+1}^{\infty} C_a n^{-10}\rho_a^{t'-I_n}\left|\zeta_{t-t',k}\right| \\ &\leq \sum_{t'=1}^{\infty} C_a n^{-10}\rho_a^{t'}\left|\zeta_{t-I_n-t',k}\right|.\end{aligned}$$

Hence

$$\max_{1\leq k\leq k_n}\max_{1\leq\tau\leq n}\left|\sum_{t=1}^{\tau}\xi_{tk} - \sum_{t=1}^{\tau}\sum_{t'=0}^{I_n} a_{t'k}\zeta_{t-t',k}\right| \leq \max_{1\leq k\leq k_n}\max_{1\leq\tau\leq n}\sum_{t'=1}^{\infty} C_a n^{-9}\rho_a^{t'}\left|\zeta_{t-I_n-t',k}\right|.$$

Denote $W_{tk} = \sum_{t'=1}^{\infty}\rho_a^{t'}\left|\zeta_{t-I_n-t',k}\right|$, by noticing that $\sup_{t,k}\mathrm{E}\left|\zeta_{t,k}\right|^{r_0} < \infty$,

$$\left\|W_{tk}\right\|_{r_0} \leq \sum_{t'=1}^{\infty}\rho_a^{t'}\left\|\zeta_{t-I_n-t',k}\right\|_{r_0} < \infty.$$

which implies $\mathrm{E}\left|W_{tk}\right|^{r_0} < K$ for some $K > 0, 1 \leq t \leq n, 1 \leq k \leq k_n$. Note that $k_n = \mathcal{O}(n^{\omega})$ in Assumption (A4), thus for β_1 in Assumption (A5)

$$\mathbb{P}\left(C_a n^{-9}\max_{1\leq k\leq k_n}\max_{1\leq\tau\leq n} W_{tk} > M n^{\beta_1}\right) \leq n k_n \frac{C_a^{r_0} K}{M^{r_0}} n^{-r_0(\beta_1+9)} < \frac{C_a^{r_0} K}{M^{r_0}} n^{-r_0(\beta_1+9)+1+\omega}.$$

Therefore

$$\mathbb{P}\left(\max_{1\le k\le k_n}\max_{1\le\tau\le n}\left|\sum_{t=1}^{\tau}\xi_{tk}-\sum_{t=1}^{\tau}\sum_{t'=0}^{I_n}a_{t'k}\zeta_{t-t',k}\right|>Mn^{\beta_1}\right)<\frac{C_a^{r_0}K}{M^{r_0}}n^{-r_0(\beta_1+9)+1+\omega}.$$

Next denote $U_{tk}=\sum_{t'=I_n+1}^{\infty}a_{t'k}Z_{t-t',k,\zeta}$, then $U_{tk}\sim N\left(0,\sum_{t'=I_n+1}^{\infty}a_{t'k}^2\right),1\le k\le k_n$. It is clear that

$$\max_{1\le k\le k_n}\max_{1\le\tau\le n}\left|\sum_{t=1}^{\tau}\sum_{t'=I_n+1}^{\infty}a_{t'k}Z_{t-t',k,\zeta}\right|\le n\max_{1\le k\le k_n}\max_{1\le t\le n}|U_{tk}|.$$

Note that $\sum_{t'=I_n+1}^{\infty}a_{t'k}^2<Cn^{-20}$ for some $C>0$, then one can obtain

$$\mathbb{P}\left(n\max_{1\le k\le k_n}\max_{1\le t\le n}|U_{tk}|>Mn^{\beta_1}\right)\le nk_n\frac{Cn^{-20}}{M^2}n^{-2(\beta_1-1)}\le\frac{C}{M^2}n^{-17-2\beta_1+\omega},$$

which implies that

$$\mathbb{P}\left(\max_{1\le k\le k_n}\max_{1\le\tau\le n}\left|\sum_{t=1}^{\tau}\sum_{t'=I_n+1}^{\infty}a_{t'k}Z_{t-t',k,\zeta}\right|>Mn^{\beta_1}\right)\le\frac{C}{M^2}n^{-17-2\beta_1+\omega}.$$

Now Assumption (A5) entails that

$$\mathbb{P}\left\{\max_{1\le k\le k_n}\max_{-I_n+1\le\tau\le n}\left|\sum_{t=-I_n+1}^{\tau}\zeta_{tk}-\sum_{t=-I_n+1}^{\tau}Z_{tk,\zeta}\right|>n^{\beta_1}\right\}<C_1n^{-\gamma_1}.$$

Then for $0\le t'\le I_n,1\le t\le n,-I_n+1\le t-t'\le n$

$$\begin{aligned}
&\mathbb{P}\left\{\max_{1\le k\le k_n}\max_{1\le\tau\le n}\left|\sum_{t=1}^{\tau}\sum_{t'=0}^{I_n}a_{t'k}\left(\zeta_{t-t',k}-Z_{t-t',k,\zeta}\right)\right|>2Mn^{\beta_1}\right\}\\
=\ &\mathbb{P}\left\{\max_{1\le k\le k_n}\max_{1\le\tau\le n}\left|\sum_{t'=0}^{I_n}a_{t'k}\sum_{t=1}^{\tau}\left(\zeta_{t-t',k}-Z_{t-t',k,\zeta}\right)\right|>2Mn^{\beta_1}\right\}\\
\le\ &\mathbb{P}\left\{\max_{1\le k\le k_n}\sum_{t'=0}^{I_n}|a_{t'k}|\max_{1\le\tau\le n}\left|\sum_{t=1}^{\tau}\left(\zeta_{t-t',k}-Z_{t-t',k,\zeta}\right)\right|>2Mn^{\beta_1}\right\}
\end{aligned}$$

$$\begin{aligned}
&\leq \mathbb{P}\left\{M \max_{1\leq k\leq k_n} \max_{0\leq t'\leq I_n} \max_{1\leq \tau\leq n} \left|\sum_{t=1}^{\tau}\left(\zeta_{t-t',k} - Z_{t-t',k,\zeta}\right)\right| > 2Mn^{\beta_1}\right\} \\
&= \mathbb{P}\left\{\max_{1\leq k\leq k_n} \max_{0\leq t'\leq I_n} \max_{1\leq \tau\leq n} \left|\sum_{t=1-t'}^{\tau-t'}\left(\zeta_{tk} - Z_{tk,\zeta}\right)\right| > 2n^{\beta_1}\right\} \\
&\leq \mathbb{P}\left\{\max_{1\leq k\leq k_n} \max_{0\leq t'\leq I_n} \max_{1\leq \tau\leq n} \left|\left(\sum_{t=-I_n+1}^{\tau-t'} - \sum_{t=-I_n+1}^{1-t'}\right)\left(\zeta_{tk} - Z_{tk,\zeta}\right)\right| > 2n^{\beta_1}\right\} \\
&\leq \mathbb{P}\left\{2\max_{1\leq k\leq k_n} \max_{-I_n+1\leq \tau\leq n} \left|\sum_{t=-I_n+1}^{\tau}\left(\zeta_{tk} - Z_{tk,\zeta}\right)\right| > 2n^{\beta_1}\right\} < C_1 n^{-\gamma_1}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\mathbb{P}\left[\max_{1\leq k\leq k_n} \max_{1\leq \tau\leq n} \left|\sum_{t=1}^{\tau}\xi_{tk} - \sum_{t=1}^{\tau} Z_{tk,\xi}\right| > 3Mn^{\beta_1}\right] \\
\leq\ &\mathbb{P}\left[\left\{\max_{1\leq k\leq k_n} \max_{1\leq \tau\leq n} \left|\sum_{t=1}^{\tau}\xi_{tk} - \sum_{t=1}^{\tau}\sum_{t'=0}^{I_n} a_{t'k}\zeta_{t-t',k}\right| + \max_{1\leq k\leq k_n} \max_{1\leq \tau\leq n} \left|\sum_{t=1}^{\tau}\sum_{t'=0}^{I_n} a_{t'k}\left(\zeta_{t-t',k} - Z_{t-t',k,\zeta}\right)\right|\right.\right. \\
&\left.\left.+ \max_{1\leq k\leq k_n} \max_{1\leq \tau\leq n} \left|\sum_{t=1}^{\tau}\sum_{t'=I_n+1}^{\infty} a_{t'k}Z_{t-t',k,\zeta}\right|\right\} > 3Mn^{\beta_1}\right] \\
\leq\ &\mathbb{P}\left[\max_{1\leq k\leq k_n} \max_{1\leq \tau\leq n} \left|\sum_{t=1}^{\tau}\xi_{tk} - \sum_{t=1}^{\tau}\sum_{t'=0}^{I_n} a_{t'k}\zeta_{t-t',k}\right| > Mn^{\beta_1}\right] \\
&+\mathbb{P}\left[\max_{1\leq k\leq k_n} \max_{1\leq \tau\leq n} \left|\sum_{t=1}^{\tau}\sum_{t'=0}^{I_n} a_{t'k}\left(\zeta_{t-t',k} - Z_{t-t',k,\zeta}\right)\right| > Mn^{\beta_1}\right] \\
&+\mathbb{P}\left[\max_{1\leq k\leq k_n} \max_{1\leq \tau\leq n} \left|\sum_{t=1}^{\tau}\sum_{t'=I_n+1}^{\infty} a_{t'k}Z_{t-t',k,\zeta}\right| > Mn^{\beta_1}\right] \\
\leq\ &\frac{C_a^{r_0}K}{M^{r_0}}n^{-r_0(\beta_1+9)+1+\omega} + \frac{C}{M^2}n^{-17-2\beta_1+\omega} + C_1 n^{-\gamma_1} < C_{r_0}n^{-\gamma_1}.
\end{aligned}$$

Denote $C_3 = 3M$ and $Z_{tk,\xi} = \sum_{t'=0}^{\infty} a_{t'k}Z_{t-t',k,\zeta}, 1 \leq t \leq n, 1 \leq k \leq k_n$, then $\{Z_{tk,\xi}\}_{t=1,k=1}^{\infty}$ are $N(0,1)$ variables with $\text{Cov}\left(Z_{tk,\xi}, Z_{t+h,k,\xi}\right) = \gamma_k(h)$, thus

$$\mathbb{P}\left\{\max_{1\leq k\leq k_n} \max_{1\leq \tau\leq n} \left|\sum_{t=1}^{\tau}\xi_{tk} - \sum_{t=1}^{\tau} Z_{tk,\xi}\right| > C_3 n^{\beta_1}\right\} < C_4 n^{-\gamma_1}.$$

The proof is completed.

Lemma S.5. *Under Assumptions (A2), (A3), (A6), as* $N \to \infty$

$$\max_{1\le t\le n}\max_{1\le \ell\le J_s+p}\left|N^{-1}\sum_{j=1}^{N}B_{\ell,p}\left(j/N\right)\sigma\left(j/N\right)Z_{tj,\varepsilon}\right| = \mathcal{O}_{a.s.}\left(N^{-1/2}J_s^{-1/2}\log^{1/2}N\right), \tag{S.7}$$

where $Z_{tj,\varepsilon}$, $1\le t\le n$, $1\le j\le N$, *are i.i.d* $N\left(0,1\right)$ *random variables.*

Proof. We apply Lemma S.1 to obtain uniform bound for Gaussian variables $N^{-1}\sum_{j=1}^{N}B_{\ell,p}\left(j/N\right)\sigma\left(j/N\right)Z_{tj,\varepsilon}$, $1\le t\le n, 1\le \ell\le J_s+p$ with mean 0, variance $N^{-1}\left\|B_{\ell,p}\sigma\right\|_{2,N}^2\le CN^{-1}J_s^{-1}$. Lemma S.1 implies that

$$\begin{aligned}\max_{1\le t\le n}\max_{1\le \ell\le J_s+p}\left|N^{-1}\sum_{j=1}^{N}B_{\ell,p}\left(j/N\right)\sigma\left(j/N\right)Z_{tj,\varepsilon}\right| &= \mathcal{O}_{a.s.}\left\{N^{-1/2}J_s^{-1/2}\log^{1/2}\left(J_s+p\right)n\right\}\\ &= \mathcal{O}_{a.s.}\left(N^{-1/2}J_s^{-1/2}\log^{1/2}N\right),\end{aligned} \tag{S.8}$$

where the last step follows from Assumptions (A3)and (A6) on the order of J_s and n relative to N. The lemma is proved.

Lemma S.6. *Under Assumptions (A2)-(A6), for* $h\in\mathbb{N}$, *as* $N\to\infty$

$$\max_{\substack{1\le k\le k_n\\ 1\le \ell\le J_s+p}}\left|n^{-1}N^{-1}\sum_{t=1}^{n-h}Z_{tk,\xi}\left\{\sum_{j=1}^{N}B_{\ell,p}\left(\frac{j}{N}\right)\sigma\left(\frac{j}{N}\right)\left(\varepsilon_{tj}-Z_{tj,\varepsilon}\right)\right\}\right| = \mathcal{O}_{a.s.}\left(n^{-1/2}N^{\beta_2-1}\log^{1/2}N\right),$$

where $0<\beta_2<1/2$.

Proof.Under Assumption (A5), one has

$$\max_{1\le j\le N}\max_{1\le t\le n}\left|N^{-1}\sum_{m=1}^{j}\left(\varepsilon_{tm}-Z_{tm,\varepsilon}\right)\right| = \mathcal{O}_{a.s.}\left(N^{\beta_2-1}\right).$$

Next, we denote the following sequences:

$$Q_{1,lk} = (nN)^{-1}\sum_{t=1}^{n-h}Z_{tk,\xi}\left[\sum_{j=1}^{N-1}\left\{B_{\ell,p}\left(\frac{j}{N}\right)\sigma\left(\frac{j}{N}\right)-B_{\ell,p}\left(\frac{j+1}{N}\right)\sigma\left(\frac{j+1}{N}\right)\right\}\sum_{m=1}^{j}\left(\varepsilon_{tm}-Z_{tm,\varepsilon}\right)\right],$$

$$Q_{2,lk} = (nN)^{-1}\sum_{t=1}^{n-h} Z_{tk,\xi}\left\{B_{\ell,p}(1)\,\sigma(1)\sum_{m=1}^{N}(\varepsilon_{tm} - Z_{tm,\varepsilon})\right\}.$$

Further denote the σ-field $\mathcal{F}_{\varepsilon} = \sigma\{\varepsilon_{tj}, t, j = 1, 2, \dots\}$ and

$$A_{lt} = \sum_{j=1}^{N-1}\left\{B_{\ell,p}\left(\frac{j}{N}\right)\sigma\left(\frac{j}{N}\right) - B_{\ell,p}\left(\frac{j+1}{N}\right)\sigma\left(\frac{j+1}{N}\right)\right\}N^{-1}\sum_{m=1}^{j}(\varepsilon_{tm} - Z_{tm,\varepsilon}), \tag{S.9}$$

then for $1 \le k \le k_n, 1 \le \ell \le J_s + p$, $Q_{1,lk}|\mathcal{F}_{\varepsilon} =_D N\left(0, \sigma^2_{1,lk}\right)$, in which

$$\begin{aligned}\sigma^2_{1,lk} &= n^{-2}\sum_{t=1}^{n-h} A^2_{lt} + 2n^{-2}\sum_{t<t'} A_{lt}A_{lt'}\,\mathrm{E}\left(Z_{tk,\xi}Z_{t'k,\xi}\right)\\ &= n^{-2}\sum_{t=1}^{n-h} A^2_{lt} + 2n^{-2}\sum_{t<t'} A_{lt}A_{lt'}\gamma_k\left(|t-t'|\right).\end{aligned}$$

Applying Lemma S.1, for any $a > 2$

$$\mathbb{P}\left[\max_{1\le k\le k_n}\max_{1\le \ell\le J_s+p}|Q_{1,lk}|/\sigma_{1,lk} > a\{\log k_n(J_s+p)\}^{1/2}\middle|\mathcal{F}_{\varepsilon}\right] < 2\{k_n(J_s+p)\}^{1-a^2/2},$$

and hence

$$\mathbb{P}\left[\max_{1\le k\le k_n}\max_{1\le \ell\le J_s+p}|Q_{1,lk}|/\sigma_{1,lk} > a\{\log k_n(J_s+p)\}^{1/2}\right] < 2\{k_n(J_s+p)\}^{1-a^2/2}.$$

Taking enough a, one concludes with Borel-Cantelli Lemma that

$$\max_{1\le k\le k_n}\max_{1\le \ell\le J_s+p}|Q_{1,lk}|/\sigma_{1,lk} = \mathcal{O}_{a.s.}\left(\{\log k_n(J_s+p)\}^{1/2}\right) = \mathcal{O}_{a.s.}\left(\log^{1/2} N\right).$$

Next, the B spline basis satisfies

$$\left|B_{\ell,p}\left(\frac{j}{N}\right) - B_{\ell,p}\left(\frac{j+1}{N}\right)\right| \le N^{-1}\|B_{\ell,p}\|_{0,1} \le CJ_sN^{-1}$$

uniformly over $1 \le j \le N$ and $1 \le \ell \le J_s + p$, while Assumptions (A2) and (A6) imply that $J_sN^{-1} \sim N^{\gamma}d_NN^{-1} \sim N^{\gamma-1}d_N \gg N^{-\upsilon}$, hence

$$\left|\sigma\left(\frac{j}{N}\right) - \sigma\left(\frac{j+1}{N}\right)\right| \le N^{-\upsilon}\|\sigma\|_{0,\upsilon} \le CJ_sN^{-1}$$

uniformly over $1 \le j \le N$. It then follows that

$$\max_{\substack{1\le\ell\le J_s+p \\ 1\le t\le n}} |A_{lt}| \le \left\{ \max_{1\le j\le N} \max_{1\le t\le n} \left| N^{-1} \sum_{m=1}^{j} (\varepsilon_{tm} - Z_{tm,\varepsilon}) \right| \right\} \left\{ CNJ_s^{-1} \times J_s N^{-1} \right\} = \mathcal{O}_{a.s.}\left(N^{\beta_2-1}\right). \tag{S.10}$$

then one has

$$\begin{aligned}
\sigma_{1,lk}^2 &\le \mathcal{O}_{a.s.}\left(n^{-1}N^{2\beta_2-2}\right) + 2n^{-2}\sum_{t<t'} |\gamma_k(|t-t'|)| \times \mathcal{O}_{a.s.}\left(N^{2\beta_2-2}\right) \\
&= \mathcal{O}_{a.s.}\left(n^{-1}N^{2\beta_2-2}\right) + 2n^{-2}\sum_{l=1}^{n-h-1} (n-h-l)|\gamma_k(l)| \times \mathcal{O}_{a.s.}\left(N^{2\beta_2-2}\right) \\
&= \mathcal{O}_{a.s.}\left(n^{-1}N^{2\beta_2-2}\right) + 2n^{-2}(n-h)\sum_{l=1}^{n-h-1} \left(1-\frac{l}{n-h}\right)|\gamma_k(l)| \times \mathcal{O}_{a.s.}\left(N^{2\beta_2-2}\right)
\end{aligned}$$

$$\le \mathcal{O}_{a.s.}\left(n^{-1}N^{2\beta_2-2}\right) + 2n^{-2}(n-h)\sum_{l=1}^{\infty} |\gamma_k(l)| \times \mathcal{O}_{a.s.}\left(N^{2\beta_2-2}\right) = \mathcal{O}_{a.s.}\left(n^{-1}N^{2\beta_2-2}\right).$$

Putting together one obtains that

$$\max_{1\le k\le k_n} \max_{1\le\ell\le J_s+p} |Q_{1,lk}| = \mathcal{O}_{a.s.}\left(n^{-1/2}N^{\beta_2-1}\log^{1/2} N\right).$$

Similarly, $\max_{1\le k\le k_n} \max_{1\le\ell\le J_s+p} |Q_{2,lk}| = \mathcal{O}_{a.s.}\left(n^{-1/2}N^{\beta_2-1}\log^{1/2} N\right)$. Finally, the Lemma is proved by noticing that

$$\left| (nN)^{-1} \sum_{t=1}^{n-h} Z_{tk,\xi} \left\{ \sum_{j=1}^{N} B_{\ell,p}\left(\frac{j}{N}\right) \sigma(j/N)(\varepsilon_{tj} - Z_{tj,\varepsilon}) \right\} \right| \le |Q_{1,lk}| + |Q_{2,lk}|.$$

Lemma S.7. *Under Assumptions (A2), (A3), (A5), (A6), for $h \in \mathbb{N}$, as $N \to \infty$*

$$\max_{1\le t\le n} \max_{1\le\ell\le J_s+p} \left| N^{-1} \sum_{j=1}^{N} B_{\ell,p}\left(\frac{j}{N}\right) \sigma\left(\frac{j}{N}\right) (\varepsilon_{tj} - Z_{tj,\varepsilon}) \right| = \mathcal{O}_{a.s.}\left(N^{\beta_2-1}\right),$$

$$\max_{1\le t\le n} \|\tilde{\varepsilon}_t\|_\infty = \max_{1\le t\le n} \left\| N^{-1}\mathbf{B}(x)^\top \mathbf{V}_{n,p}^{-1} \mathbf{B}^T \boldsymbol{\varepsilon}_t \right\|_\infty = \mathcal{O}_{a.s}\left\{ J_s N^{\beta_2-1} + J_s^{1/2} N^{-1/2} (\log N)^{1/2} \right\}. \tag{S.11}$$

PROOF. Note that

$$
\begin{aligned}
& N^{-1}\sum_{j=1}^{N} B_{\ell,p}\left(\frac{j}{N}\right)\sigma\left(\frac{j}{N}\right)\left(\varepsilon_{tj}-Z_{tj,\varepsilon}\right) \\
= & \sum_{j=1}^{N-1}\left\{B_{\ell,p}\left(\frac{j}{N}\right)\sigma\left(\frac{j}{N}\right)-B_{\ell,p}\left(\frac{j+1}{N}\right)\sigma\left(\frac{j+1}{N}\right)\right\}N^{-1}\sum_{m=1}^{j}\left(\varepsilon_{tm}-Z_{tm,\varepsilon}\right) \\
& +B_{\ell,p}\left(1\right)\sigma\left(1\right)N^{-1}\sum_{m=1}^{N}\left(\varepsilon_{tm}-Z_{tm,\varepsilon}\right) \\
= & A_{lt}+B_{\ell,p}\left(1\right)\sigma\left(1\right)N^{-1}\sum_{m=1}^{N}\left(\varepsilon_{tm}-Z_{tm,\varepsilon}\right),
\end{aligned}
$$

where A_{lt} is defined in (S.9). According to (S.10) in Lemma S.6 and Assumption (A5), one obtains

$$
\begin{aligned}
& \max_{1\le t\le n}\max_{1\le \ell\le J_s+p}\left|N^{-1}\sum_{j=1}^{N} B_{\ell,p}\left(\frac{j}{N}\right)\sigma\left(\frac{j}{N}\right)\left(\varepsilon_{tj}-Z_{tj,\varepsilon}\right)\right| \\
\le & \max_{1\le t\le n}\max_{1\le \ell\le J_s+p}\left|A_{lt}\right|+\mathcal{O}_{a.s.}\left(N^{\beta_2-1}\right)=\mathcal{O}_{a.s.}\left(N^{\beta_2-1}\right). \qquad \text{(S.12)}
\end{aligned}
$$

Then by noting that (S.7) in Lemma S.5 and (S.12), one has

$$
\begin{aligned}
\max_{1\le t\le n}\left\|N^{-1}\mathbf{B}^{T}\boldsymbol{\varepsilon}_t\right\|_\infty = & \max_{1\le t\le n}\max_{1\le \ell\le J_s+p}\left|N^{-1}\sum_{j=1}^{N} B_{\ell,p}\left(\frac{j}{N}\right)\sigma\left(\frac{j}{N}\right)\varepsilon_{tj}\right| \\
\le & \max_{\substack{1\le t\le n\\ 1\le \ell\le J_s+p}}\left|N^{-1}\sum_{j=1}^{N} B_{\ell,p}\left(\frac{j}{N}\right)\sigma\left(\frac{j}{N}\right)\left(\varepsilon_{tj}-Z_{tj,\varepsilon}\right)\right| \\
& +\max_{\substack{1\le t\le n\\ 1\le \ell\le J_s+p}}\left|N^{-1}\sum_{j=1}^{N} B_{\ell,p}\left(\frac{j}{N}\right)\sigma\left(\frac{j}{N}\right)Z_{tj,\varepsilon}\right| \\
\le & \mathcal{O}_{a.s.}\left(N^{\beta_2-1}+N^{-1/2}J_s^{-1/2}\log^{1/2}N\right).
\end{aligned}
$$

Therefore, by recalling $\tilde{\varepsilon}_t(x)$ in (S.4) and the result in equation (S.1), one obtains

$$
\max_{1\le t\le n}\left\|\tilde{\varepsilon}_t\right\|_\infty=\max_{1\le t\le n}\left\|N^{-1}\mathbf{B}(x)^{\top}\mathbf{V}_{n,p}^{-1}\mathbf{B}^{T}\boldsymbol{\varepsilon}_t\right\|_\infty=\mathcal{O}_{a.s}\left\{J_sN^{\beta_2-1}+J_s^{1/2}N^{-1/2}(\log N)^{1/2}\right\}.
$$

Lemma S.8. *Under Assumptions (A2)-(A6), for $h \in \mathbb{N}$, as $N \to \infty$*

$$\begin{aligned}
&\max_{\substack{1\le k\le k_n\\ 1\le \ell\le J_s+p}} \left|(nN)^{-1}\sum_{t=1}^{n-h}(\xi_{tk}-Z_{tk,\xi})\sum_{j=1}^{N}B_{\ell,p}\left(\frac{j}{N}\right)\sigma\left(\frac{j}{N}\right)Z_{tj,\varepsilon}\right| \\
= \quad & \mathcal{O}_{a.s.}\left(n^{\beta_1-1/2}N^{-1/2}J_s^{-1/2}\log^{1/2}N\right),
\end{aligned}$$

where $0<\beta_1<1/2$.

Proof. According to Lemma S.4,

$$\max_{1\le k\le k_n}\max_{1\le\tau\le n}\left|\sum_{t=1}^{\tau}(\xi_{tk}-Z_{tk,\xi})\right| = \mathcal{O}_{a.s.}\left(n^{\beta_1}\right).$$

Next, denote

$$\begin{aligned}
\mathrm{Q}_{3,lk} &= (nN)^{-1}\sum_{\tau=1}^{n-h-1}\sum_{t=1}^{\tau}(\xi_{tk}-Z_{tk,\xi})\sum_{j=1}^{N}B_{\ell,p}\left(\frac{j}{N}\right)\sigma\left(\frac{j}{N}\right)Z_{\tau j,\varepsilon}, \\
\mathrm{Q}_{4,lk} &= -(nN)^{-1}\sum_{\tau=1}^{n-h-1}\sum_{t=1}^{\tau}(\xi_{tk}-Z_{tk,\xi})\sum_{j=1}^{N}B_{\ell,p}\left(\frac{j}{N}\right)\sigma\left(\frac{j}{N}\right)Z_{(\tau+1)j,\varepsilon} \\
\mathrm{Q}_{5,lk} &= (nN)^{-1}\sum_{t=1}^{n-h}(\xi_{tk}-U_{tk,\xi})\sum_{j=1}^{N}B_{\ell,p}\left(\frac{j}{N}\right)\sigma\left(\frac{j}{N}\right)Z_{n-h,j,\varepsilon}.
\end{aligned}$$

Denote the σ-field $\mathcal{F}_\xi = \sigma\{\xi_{tk},\ t,k=1,2,\ldots\}$, then for $1\le k\le k_n, 1\le \ell\le J_s+p$, one has $\mathrm{Q}_{3,lk}\big|\,\mathcal{F}_\xi =_D N\left(0,\sigma^2_{lk,3}\right)$, where

$$\sigma^2_{lk,3} = (nN)^{-2}\sum_{\tau=1}^{n-h-1}\left\{\sum_{t=1}^{\tau}(\xi_{tk}-Z_{tk,\xi})\right\}^2\sum_{j=1}^{N}B^2_{\ell,p}\left(\frac{j}{N}\right)\sigma^2\left(\frac{j}{N}\right).$$

Similar to Lemma S.6, applying (S.6), for any $a>2$

$$\mathbb{P}\left[\max_{1\le k\le k_n}\max_{1\le\ell\le J_s+p}\left|\mathrm{Q}_{3,lk}\right|/\sigma_{lk,3} > a\left\{\log k_n(J_s+p)\right\}^{1/2}\middle|\,\mathcal{F}_\xi\right] \le \sqrt{\pi/2}\left\{k_n(J_s+p)\right\}^{1-a^2/2},$$

and hence

$$\mathbb{P}\left[\max_{1\le k\le k_n}\max_{1\le\ell\le J_s+p}\left|\mathrm{Q}_{3,lk}\right|/\sigma_{lk,3} > a\left\{\log k_n(J_s+p)\right\}^{1/2}\right] \le \sqrt{\pi/2}\left\{k_n(J_s+p)\right\}^{1-a^2/2}.$$

Taking large enough a, according to the order conditions on k_n, J_s in Assumptions (A4) and (A6), one concludes with Borel-Cantelli Lemma that

$$\max_{1\leq k\leq k_n}\max_{1\leq \ell\leq J_s+p}\left|\mathrm{Q}_{3,lk}\right|/\sigma_{lk,3}=\mathcal{O}_{a.s.}\left(\{\log k_n\left(J_s+p\right)\}^{1/2}\right)=\mathcal{O}_{a.s.}\left(\log^{1/2}N\right). \tag{S.13}$$

Noticing that $\max_{1\leq\ell\leq J_s+p}\|B_{\ell,p}\sigma\|_{2,N}^2=\mathcal{O}\left(J_s^{-1}\right)$, one has

$$\begin{aligned}\sigma_{lk,3}^2&=n^{-2}\sum_{\tau=1}^{n-h-1}\left\{\sum_{t=1}^{\tau}\left(\xi_{tk}-Z_{tk,\xi}\right)\right\}^2N^{-2}\sum_{j=1}^{N}B_{\ell,p}^2\left(\frac{j}{N}\right)\sigma^2\left(\frac{j}{N}\right)\\&\leq n\left\{\max_{1\leq k\leq k_n}\max_{1\leq\tau\leq n}\left|n^{-1}\sum_{t=1}^{\tau}\left(\xi_{tk}-Z_{tk,\xi}\right)\right|\right\}^2N^{-1}\|B_{\ell,p}\sigma\|_{2,N}^2\leq cJ_s^{-1}N^{-1}n^{2\beta_1-1}.\end{aligned} \tag{S.14}$$

Putting together the bounds in (S.13) and (S.14), one obtains that

$$\max_{1\leq k\leq k_n}\max_{1\leq\ell\leq J_s+p}\left|\mathrm{Q}_{3,lk}\right|=\mathcal{O}_{a.s.}\left(n^{\beta_1-1/2}J_s^{-1/2}N^{-1/2}\log^{1/2}N\right).$$

One can show the following similarly,

$$\max_{1\leq k\leq k_n}\max_{1\leq\ell\leq J_s+p}\left|\mathrm{Q}_{4,lk}\right|=\mathcal{O}_{a.s.}\left(n^{\beta_1-1/2}J_s^{-1/2}N^{-1/2}\log^{1/2}N\right),$$

$$\max_{1\leq k\leq k_n}\max_{1\leq\ell\leq J_s+p}\left|\mathrm{Q}_{5,lk}\right|=\mathcal{O}_{a.s.}\left(n^{\beta_1-1/2}J_s^{-1/2}N^{-1/2}\log^{1/2}N\right).$$

Therefore, the lemma holds by noticing that

$$\begin{aligned}&\max_{1\leq k\leq k_n}\max_{1\leq\ell\leq J_s+p}\left|(nN)^{-1}\sum_{t=1}^{n-h}\left(\xi_{tk}-Z_{tk,\xi}\right)\sum_{j=1}^{N}B_{\ell,p}\left(\frac{j}{N}\right)\sigma\left(\frac{j}{N}\right)Z_{tj,\varepsilon}\right|\\&\quad\leq\max_{1\leq k\leq k_n}\max_{1\leq\ell\leq J_s+p}\left|\mathrm{Q}_{3,lk}\right|+\max_{1\leq k\leq k_n}\max_{1\leq\ell\leq J_s+p}\left|\mathrm{Q}_{4,lk}\right|+\max_{1\leq k\leq k_n}\max_{1\leq\ell\leq J_s+p}\left|\mathrm{Q}_{5,lk}\right|.\end{aligned}$$

Lemma S.9. *Under Assumptions (A2)-(A6), for $h\in\mathbb{N}$, as $N\to\infty$*

$$\begin{aligned}&\max_{1\leq k\leq k_n}\max_{1\leq\ell\leq J_s+p}\left|(nN)^{-1}\sum_{t=1}^{n-h}\left(\xi_{tk}-Z_{tk,\xi}\right)\left\{\sum_{j=1}^{N}B_{\ell,p}\left(\frac{j}{N}\right)\sigma\left(\frac{j}{N}\right)\left(\varepsilon_{tj}-Z_{tj,\varepsilon}\right)\right\}\right|\\&=\ \mathcal{O}_{a.s.}\left(n^{\beta_1}N^{\beta_2-1}\right).\end{aligned}$$

PROOF. By noticing that

$$\max_{1\le\tau\le n}\max_{1\le k\le k_n}\left|n^{-1}\sum_{t=1}^{\tau}\left(\xi_{tk}-Z_{tk,\xi}\right)\right|=\mathcal{O}_{a.s.}\left(n^{\beta_1-1}\right),$$

$$\max_{1\le t\le n}\max_{1\le \ell\le J_s+p}\left|N^{-1}\sum_{j=1}^{N}B_{\ell,p}\left(\frac{j}{N}\right)\sigma\left(\frac{j}{N}\right)\left(\varepsilon_{tj}-Z_{tj,\varepsilon}\right)\right|=\mathcal{O}_{a.s.}\left(N^{\beta_2-1}\right),$$

one has

$$\max_{1\le k\le k_n}\max_{1\le \ell\le J_s+p}\left|(nN)^{-1}\sum_{t=1}^{n-h}\left(\xi_{tk}-Z_{tk,\xi}\right)\left\{\sum_{j=1}^{N}B_{\ell,p}\left(\frac{j}{N}\right)\sigma\left(\frac{j}{N}\right)\left(\varepsilon_{tj}-Z_{tj,\varepsilon}\right)\right\}\right|$$
$$\le\max_{1\le k\le k_n}n^{-1}\sum_{t=1}^{n-h}\left|\xi_{tk}-Z_{tk,\xi}\right|\max_{1\le \ell\le J_s+p}\max_{1\le t\le n}\left|N^{-1}\sum_{j=1}^{N}B_{\ell,p}\left(\frac{j}{N}\right)\sigma\left(\frac{j}{N}\right)\left(\varepsilon_{tj}-Z_{tj,\varepsilon}\right)\right|$$
$$\le\left\{\max_{1\le k\le k_n}n^{-1}\left|\xi_{1k}-Z_{1k,\xi}\right|+\max_{1\le k\le k_n}\sum_{t=2}^{n-h}n^{-1}\left|\sum_{\tau=1}^{t}\left(\xi_{\tau k}-Z_{\tau k,\xi}\right)-\sum_{\tau=1}^{t-1}\left(\xi_{\tau k}-Z_{\tau k,\xi}\right)\right|\right\}$$
$$\times\mathcal{O}_{a.s.}\left(N^{\beta_2-1}\right)$$

$$\le\left\{\mathcal{O}_{a.s.}\left(n^{\beta_1-1}\right)+\max_{1\le k\le k_n}\sum_{t=2}^{n-h}2\max_{1\le t\le n}\left|n^{-1}\sum_{\tau=1}^{t}\left(\xi_{\tau k}-Z_{\tau k,\xi}\right)\right|\right\}\times\mathcal{O}_{a.s.}\left(N^{\beta_2-1}\right)$$
$$\le\left\{\mathcal{O}_{a.s.}\left(n^{\beta_1-1}\right)+n\mathcal{O}_{a.s.}\left(n^{\beta_1-1}\right)\right\}\times\mathcal{O}_{a.s.}\left(N^{\beta_2-1}\right)=\mathcal{O}_{a.s.}\left(n^{\beta_1}N^{\beta_2-1}\right)$$

The lemma holds.

Lemma S.10. *Under Assumptions (A1), (A3)–(A6), as $N\to\infty$*

$$\max_{1\le t\le n}\left\|\tilde{\eta}_t-\eta_t\right\|_\infty=\mathcal{O}_{a.s.}\left\{J_s^{-p^*}\left(n\log n\right)^{2/r_0}\right\},\tag{S.15}$$

$$\max_{1\le t\le n}\left\|\tilde{\chi}_t-\chi_t\right\|_\infty=\mathcal{O}_{a.s.}\left\{J_s^{-p^*}\left(n\log n\right)^{2/r_0}\right\},\tag{S.16}$$

$$\max_{1\le t\le n}\left\|\chi_t\right\|_\infty=\mathcal{O}_{a.s.}\left\{\left(n\log n\right)^{2/r_0}\right\}.\tag{S.17}$$

PROOF. For any $k=1,2,\ldots$, let $\boldsymbol{\phi}_k=\left(\phi_k\left(1/N\right),\ldots,\phi_k\left(N/N\right)\right)^\top$, and denote $\tilde{\phi}_k\left(x\right)=N^{-1}\mathbf{B}\left(x\right)^\top\mathbf{V}_{n,p}^{-1}\mathbf{B}^\top\boldsymbol{\phi}_k$. According to (S.3), $\tilde{\eta}_t\left(x\right)=\tilde{m}\left(x\right)+$

$\sum_{k=1}^{\infty}\xi_{tk}\tilde{\phi}_k(x)$, therefore,

$$\tilde{\eta}_t(x)-\eta_t(x)=\tilde{m}(x)-m(x)+\sum_{k=1}^{\infty}\xi_{tk}\left\{\tilde{\phi}_k(x)-\phi_k(x)\right\}.$$

By Lemma A.4 of Cao et al. (2012), there exists a constant $C_{q,\mu}>0$, such that

$$\|\tilde{m}-m\|_{\infty}\leq C_{q,\mu}\|m\|_{q,\mu}J_s^{-p^*}, \tag{S.18}$$

$$\left\|\tilde{\phi}_k-\phi_k\right\|_{\infty}\leq C_{q,\mu}\|\phi_k\|_{q,\mu}J_s^{-p^*},\quad k\geq 1 \tag{S.19}$$

Thus, from (S.18) and (S.19), by Assumption (A4) one obtains

$$\|\tilde{\eta}_t-\eta_t\|_{\infty}\leq\|\tilde{m}-m\|_{\infty}+\sum_{k=1}^{\infty}|\xi_{tk}|\left\|\tilde{\phi}_k-\phi_k\right\|_{\infty}\leq C_{q,\mu}W_tJ_s^{-p^*},$$

where $W_t=\|m\|_{q,\mu}+\sum_{k=1}^{\infty}|\xi_{tk}|\|\phi_k\|_{q,\mu}$, $t=1,\ldots,n$, are nonnegative random variables with finite fourth moment according to Assumption (A5). Therefore

$$\mathbb{P}\left\{\max_{1\leq t\leq n}W_t>(n\log n)^{2/r_0}\right\}\leq n\frac{\mathrm{E}W_t^{r_0}}{(n\log n)^2}=\mathrm{E}W_t^{r_0}n^{-1}(\log n)^{-2},$$

thus,

$$\sum_{n=1}^{\infty}\mathbb{P}\left\{\max_{1\leq t\leq n}W_t>(n\log n)^{2/r_0}\right\}\leq\mathrm{E}W_1^{r_0}\sum_{n=1}^{\infty}n^{-1}(\log n)^{-2}<+\infty,$$

so $\max_{1\leq t\leq n}W_t=\mathcal{O}_{a.s.}\left\{(n\log n)^{2/r_0}\right\}$ and (S.15) is proved. Similarly, one obtains that $\max_{1\leq t\leq n}\|\chi_t\|_{\infty}=\mathcal{O}_{a.s.}\left\{(n\log n)^{2/r_0}\right\}$ and $\max_{1\leq t\leq n}\|\tilde{\chi}_t-\chi_t\|_{\infty}=\mathcal{O}_{a.s.}\left\{J_s^{-p^*}(n\log n)^{2/r_0}\right\}$. Lemma S.10 holds consequently.

Lemma S.11. *Under Assumptions (A1)-(A6), as $N\to\infty$*

$$\max_{1\leq t\leq n}\|\hat{\eta}_t-\eta_t\|_{\infty}=\mathcal{O}_p\left\{J_s^{-p^*}(n\log n)^{2/r_0}+J_s^{1/2}N^{-1/2}(\log N)^{1/2}+J_sN^{\beta_2-1}\right\}, \tag{S.20}$$

$$\max_{1\leq t\leq n}\|\hat{\chi}_t-\chi_t\|_{\infty}=\mathcal{O}_p\left\{J_s^{-p^*}(n\log n)^{2/r_0}+J_s^{1/2}N^{-1/2}(\log N)^{1/2}+J_sN^{\beta_2-1}+n^{-1/2}\right\}. \tag{S.21}$$

PROOF. (S.20) follows directly from Lemmas S.10, (S.11), (S.2) and (S.5).

By simple algebra, one can obtain

$$\begin{aligned} \left\|\hat{\chi}_t-\chi_t\right\|_{\infty} &= \left\|\hat{\eta}_t-\eta_t-n^{-1}\sum_{i=1}^{n}\left(\hat{\eta}_i-\eta_i\right)-n^{-1}\sum_{i=1}^{n}\chi_i\right\|_{\infty} \\ &\leq 2\left\|\hat{\eta}_t-\eta_t\right\|_{\infty}+\left\|n^{-1}\sum_{t=1}^{n}\chi_t\right\|_{\infty} \\ &= 2\left\|\hat{\eta}_t-\eta_t\right\|_{\infty}+\left\|n^{-1}\sum_{t=1}^{n}\sum_{k=1}^{\infty}\xi_{tk}\phi_k\left(x\right)\right\|_{\infty} \\ &= 2\left\|\hat{\eta}_t-\eta_t\right\|_{\infty}+\left\|\sum_{k=1}^{\infty}\bar{\xi}_{\cdot k}\phi_k\left(x\right)\right\|_{\infty}, \end{aligned} \tag{S.22}$$

where $\bar{\xi}_{\cdot k}=n^{-1}\sum_{t=1}^{n}\xi_{tk}, k\geq 1$. Then one has

$$\left\|\sum_{k=1}^{\infty}\bar{\xi}_{\cdot k}\phi_k\left(x\right)\right\|_{\infty}\leq\sup_{x\in[0,1]}\sum_{k=1}^{\infty}\left|\bar{\xi}_{\cdot k}\right|\left|\phi_k\left(x\right)\right|\leq\sum_{k=1}^{\infty}\left|\bar{\xi}_{\cdot k}\right|\left\|\phi_k\right\|_{\infty}.$$

Note that under Assumption (A5')

$$\left(\mathrm{E}\left|\bar{\xi}_{\cdot k}\right|\right)^2\leq\mathrm{E}\left|\bar{\xi}_{\cdot k}\right|^2=n^{-1}+2n^{-2}\sum_{l=1}^{n-1}\left(n-l\right)\gamma_k\left(l\right)=\mathcal{O}\left(n^{-1}\right).$$

Hence $\max_{1\leq k<\infty}\mathrm{E}\left|\bar{\xi}_{\cdot k}\right|=\mathcal{O}\left(n^{-1/2}\right)$ and then $\mathrm{E}\left\|\sum_{k=1}^{\infty}\bar{\xi}_{\cdot k}\phi_k\left(x\right)\right\|_{\infty}\leq\sum_{k=1}^{\infty}\mathrm{E}\left|\bar{\xi}_{\cdot k}\right|\left\|\phi_k\right\|_{\infty}=\mathcal{O}\left(n^{-1/2}\right)$. Hence one obtains that

$$\left\|\sum_{k=1}^{\infty}\bar{\xi}_{\cdot k}\phi_k\left(x\right)\right\|_{\infty}=\mathcal{O}_p\left(n^{-1/2}\right). \tag{S.23}$$

Combining (S.22), (S.23) and (S.20), one has

$$\begin{aligned} \max_{1\leq t\leq n}\left\|\hat{\chi}_t-\chi_t\right\|_{\infty} &\leq \max_{1\leq t\leq n}\left\|\hat{\eta}_t-\eta_t\right\|_{\infty}+\mathcal{O}_p\left(n^{-1/2}\right) \\ &= \mathcal{O}_p\left\{J_s^{-p^*}\left(n\log n\right)^{2/r_0}+J_s^{1/2}N^{-1/2}\left(\log N\right)^{1/2}+J_sN^{\beta_2-1}+n^{-1/2}\right\}. \end{aligned}$$

The proof is completed.

Lemma S.12. *Under Assumptions (A1)-(A6), for $h \in \mathbb{N}$, as $N \to \infty$*

$$\sup_{x,x' \in [0,1]} \left| n^{-1} \sum_{t=1}^{n-h} \chi_{t+h}(x') \left\{ \tilde{\chi}_t(x) - \chi_t(x) \right\} \right| = \scriptstyle{\mathcal{O}}_p\left(n^{-1/2}\right).$$

Proof. According to Lemma S.10, one has

$$\sup_{x,x' \in [0,1]} \left| n^{-1} \sum_{t=1}^{n-h} \chi_{t+h}(x') \left\{ \tilde{\chi}_t(x) - \chi_t(x) \right\} \right| \leq \max_{1 \leq t \leq n} \|\chi_t\|_\infty \times \max_{1 \leq t \leq n} \|\tilde{\chi}_t - \chi_t\|_\infty$$
$$= \mathcal{O}_{a.s.}\left\{ J_s^{-p^*} (n \log n)^{4/r_0} \right\} = \scriptstyle{\mathcal{O}}_p(n^{-1/2}).$$

The proof is completed.

Lemma S.13. *Under Assumptions (A2)-(A6), for $h \in \mathbb{N}$, as $N \to \infty$*

$$\sup_{x,x' \in [0,1]} \left| n^{-1} \sum_{t=1}^{n-h} \chi_{t+h}(x') \tilde{\varepsilon}_t(x) \right| = o_p\left(n^{-1/2}\right).$$

Proof. By the definition of $\tilde{\varepsilon}_t(x)$ in (S.4), one has

$$\chi_{t+h}(x') \tilde{\varepsilon}_t(x) = N^{-1} \mathbf{B}(x)^\top \mathbf{V}_{n,p}^{-1} \mathbf{B}^\top \left\{ \chi_{t+h}(x') \sigma\left(\frac{j}{N}\right) \varepsilon_{tj} \right\}_{j=1}^{N},$$

which implies that

$$\sum_{t=1}^{n-h} \chi_{t+h}(x') \tilde{\varepsilon}_t(x) = \mathbf{B}(x)^\top \mathbf{V}_{n,p}^{-1} \left\{ \sum_{k=1}^{\infty} \phi_k(x') \frac{1}{N} \sum_{t=1}^{n-h} \sum_{j=1}^{N} B_{\ell,p}\left(\frac{j}{N}\right) \sigma\left(\frac{j}{N}\right) \xi_{t+h,k} \varepsilon_{tj} \right\}_{\ell=1}^{J_s+p}$$
$$= \mathbf{B}(x)^\top \mathbf{V}_{n,p}^{-1} \left\{ \left(\sum_{k=1}^{k_n} + \sum_{k=k_n+1}^{\infty} \right) \phi_k(x') \frac{1}{N} \sum_{t=1}^{n-h} \sum_{j=1}^{N} B_{\ell,p}\left(\frac{j}{N}\right) \sigma\left(\frac{j}{N}\right) \xi_{t+h,k} \varepsilon_{tj} \right\}_{\ell=1}^{J_s+p}.$$

First, uniformly for $1 \leq \ell \leq J_s + p, x \in [0,1]$, one has

$$\left| (nN)^{-1} \sum_{k=k_n+1}^{\infty} \phi_k(x') \sum_{t=1}^{n-h} \sum_{j=1}^{N} B_{\ell,p}\left(\frac{j}{N}\right) \sigma\left(\frac{j}{N}\right) \xi_{t+h,k} \varepsilon_{tj} \right| \leq \mathrm{D},$$

where $\mathrm{D} = \max_{1 \leq \ell \leq J_s+p} \sum_{k=k_n+1}^{\infty} \|\phi_k\|_\infty (nN)^{-1} \sum_{t=1}^{n-h} \sum_{j=1}^{N} B_{\ell,p}(j/N) \sigma\left(\frac{j}{N}\right) \left|\xi_{t+h,k}\right| |\varepsilon_{tj}|$, and

$$\begin{aligned}\mathrm{E}\,\mathrm{D} &\leq \max_{1\leq\ell\leq J_s+p}\sum_{k=k_n+1}^{\infty}\left\|\phi_k\right\|_{\infty}(nN)^{-1}\sum_{t=1}^{n-h}\sum_{j=1}^{N}B_{\ell,p}\left(\frac{j}{N}\right)\sigma\left(\frac{j}{N}\right)\mathrm{E}\left|\xi_{t+h,k}\right|\mathrm{E}\left|\varepsilon_{tj}\right| \\ &\leq cJ_s^{-1}\sum_{k=k_n+1}^{\infty}\left\|\phi_k\right\|_{\infty}.\end{aligned}$$

As Assumption (A4) guarantees that $\sum_{k=k_n+1}^{\infty}\left\|\phi_k\right\|_{\infty} \ll n^{-1/2}$, while $\left\|\mathbf{V}_{n,p}^{-1}\right\| \leq CJ_s$ for large N, one has

$$\sup_{x,x'\in[0,1]}\left|\mathbf{B}(x)^{\top}\mathbf{V}_{n,p}^{-1}\left\{\sum_{k=k_n+1}^{\infty}\phi_k\left(x'\right)\frac{1}{nN}\sum_{t=1}^{n-h}\sum_{j=1}^{N}B_{\ell,p}\left(\frac{j}{N}\right)\sigma\left(\frac{j}{N}\right)\xi_{t+h,k}\varepsilon_{tj}\right\}\right| = o_p\left(n^{-1/2}\right). \tag{S.24}$$

To bound the sum $\sum_{k=1}^{k_n}$, note that Lemma S.8 implies that

$$\begin{aligned}&\sup_{x,x'\in[0,1]}\left|\mathbf{B}(x)^{\top}\mathbf{V}_{n,p}^{-1}\left\{\sum_{k=1}^{k_n}\phi_k\left(x'\right)n^{-1}N^{-1}\sum_{t=1}^{n-h}\left(\xi_{tk}-Z_{tk,\xi}\right)\sum_{j=1}^{N}B_{\ell,p}\left(\frac{j}{N}\right)\sigma\left(\frac{j}{N}\right)Z_{tj,\varepsilon}\right\}\right| \\ &= \mathcal{O}_p\left(n^{\beta_1-1/2}N^{-1/2}J_s^{1/2}\log^{1/2}N\right).\end{aligned}$$

Note that $Z_{ik,\xi}$ and $Z_{ij,\varepsilon}$ are independent standard normal random variables. Applying similar arguments in Lemma S.6, one obtains that

$$\max_{1\leq\ell\leq J_s+p}\left|n^{-1}\sum_{t=1}^{n-h}Z_{ik,\xi}\left\{\frac{1}{N}\sum_{j=1}^{N}B_{\ell,p}\left(\frac{j}{N}\right)\sigma\left(\frac{j}{N}\right)Z_{tj,\varepsilon}\right\}\right| = \mathcal{O}_{a.s.}\left(N^{-1/2}J_s^{-1/2}n^{-1/2}\log N\right).$$

It is easy to see that

$$\left|\mathbf{B}\left(x\right)^{\top}\mathbf{V}_{n,p}^{-1}\left\{\sum_{k=1}^{k_n}\phi_k\left(x'\right)\frac{1}{nN}\sum_{t=1}^{n-h}\sum_{j=1}^{N}B_{\ell,p}\left(\frac{j}{N}\right)\sigma\left(\frac{j}{N}\right)\xi_{tk}\varepsilon_{tj}\right\}\right|$$
$$\leq\left|\mathbf{B}\left(x\right)^{\top}\mathbf{V}_{n,p}^{-1}\left\{\sum_{k=1}^{k_n}\phi_k\left(x'\right)\frac{1}{nN}\sum_{t=1}^{n-h}\left(\xi_{tk}-Z_{tk,\xi}\right)\sum_{j=1}^{N}B_{\ell,p}\left(\frac{j}{N}\right)\sigma\left(\frac{j}{N}\right)\left(\varepsilon_{tj}-Z_{tj,\varepsilon}\right)\right\}\right|$$
$$+\left|\mathbf{B}\left(x\right)^{\top}\mathbf{V}_{n,p}^{-1}\left\{\sum_{k=1}^{k_n}\phi_k\left(x'\right)\frac{1}{nN}\sum_{t=1}^{n-h}Z_{tk,\xi}\sum_{j=1}^{N}B_{\ell,p}\left(\frac{j}{N}\right)\sigma\left(\frac{j}{N}\right)\left(\varepsilon_{tj}-Z_{tj,\varepsilon}\right)\right\}\right|$$
$$+\left|\mathbf{B}\left(x\right)^{\top}\mathbf{V}_{n,p}^{-1}\left\{\sum_{k=1}^{k_n}\phi_k\left(x'\right)\frac{1}{n}\sum_{t=1}^{n-h}\left(\xi_{tk}-Z_{tk,\xi}\right)N^{-1}\sum_{j=1}^{N}B_{\ell,p}\left(\frac{j}{N}\right)\sigma\left(\frac{j}{N}\right)Z_{tj,\varepsilon}\right\}\right|$$
$$+\left|\mathbf{B}\left(x\right)^{\top}\mathbf{V}_{n,p}^{-1}\left[\sum_{k=1}^{k_n}\phi_k\left(x'\right)\frac{1}{n}\sum_{t=1}^{n-h}Z_{ik,\xi}\left\{N^{-1}\sum_{j=1}^{N}B_{\ell,p}\left(\frac{j}{N}\right)\sigma\left(\frac{j}{N}\right)Z_{tj,\varepsilon}\right\}\right]\right|.$$

Therefore, combining Lemmas S.5–S.9, one has

$$\sup_{x,x'\in[0,1]}\left|\mathbf{B}\left(x\right)^{\top}\mathbf{V}_{n,p}^{-1}\left\{\sum_{k=k_n+1}^{\infty}\phi_k\left(x'\right)\frac{1}{nN}\sum_{t=1}^{n-h}\sum_{j=1}^{N}B_{\ell,p}\left(\frac{j}{N}\right)\sigma\left(\frac{j}{N}\right)\xi_{tk}\varepsilon_{tj}\right\}\right|$$
$$=\mathcal{O}_{a.s.}\left(J_s n^{\beta_1}N^{\beta_2-1}\right)+\mathcal{O}_{a.s.}\left(J_s n^{-1/2}N^{\beta_2-1}\log^{1/2}N\right)+\mathcal{O}_{a.s.}\left(n^{\beta_1-1/2}N^{-1/2}J_s^{1/2}\log^{1/2}N\right)$$
$$+\mathcal{O}_{a.s.}\left(n^{-1/2}N^{-1/2}J_s^{1/2}\log N\right).$$

Hence, the proof is completed by noticing that

$$\sup_{x,x'\in[0,1]}\left|\frac{1}{n}\sum_{i=1}^{n-h}\chi_{t+h}\left(x'\right)\tilde{\varepsilon}_t\left(x\right)\right|$$
$$=\sup_{x,x'\in[0,1]}\left|\mathbf{B}\left(x\right)^{\top}\mathbf{V}_{n,p}^{-1}\left\{\sum_{k=1}^{\infty}\phi_k\left(x'\right)\frac{1}{nN}\sum_{t=1}^{n-h}\sum_{j=1}^{N}B_{\ell,p}\left(\frac{j}{N}\right)\sigma\left(\frac{j}{N}\right)\xi_{tk}\varepsilon_{tj}\right\}\right|$$
$$=o_p\left(n^{-1/2}\right)+\mathcal{O}_{a.s.}\left(J_s n^{-1/2}N^{\beta_2-1}\log^{1/2}N\right)+\mathcal{O}_{a.s.}\left(n^{\beta_1-1/2}N^{-1/2}J_s^{1/2}\log^{1/2}N\right)$$
$$+\mathcal{O}_{a.s.}\left(J_s n^{\beta_1}N^{\beta_2-1}\right)+\mathcal{O}_{a.s.}\left(n^{-1/2}N^{-1/2}J_s^{1/2}\log N\right)=o_p\left(n^{-1/2}\right).$$

Lemma S.14. *Under Assumptions (A2)-(A6), for $h\in\mathbb{N}$, as $N\to\infty$*

$$\sup_{x,x'\in[0,1]}\left|n^{-1}\sum_{t=1}^{n-h}\chi_{t+h}\left(x'\right)n^{-1}\sum_{i'=1}^{n}\tilde{\chi}_{i'}\left(x\right)\right|=o_p\left(n^{-1/2}\right).$$

PROOF. Notice that

$$\chi_{t+h}(x')n^{-1}\sum_{i'=1}^{n}\tilde{\chi}_{i'}(x) = N^{-1}\mathbf{B}(x)^{\top}\mathbf{V}_{n,p}^{-1}\mathbf{B}^{\top}\left\{\chi_{t+h}(x')\,n^{-1}\sum_{i'=1}^{n}\sum_{k=1}^{\infty}\xi_{i'k}\phi_k(j/N)\right\}_{j=1}^{N}$$
$$= N^{-1}\mathbf{B}(x)^{\top}\mathbf{V}_{n,p}^{-1}\mathbf{B}^{\top}\left\{\sum_{k=1}^{\infty}\xi_{t+h,k}\phi_k(x')\,n^{-1}\sum_{i'=1}^{n}\sum_{k'=1}^{\infty}\xi_{i'k'}\phi_{k'}(j/N)\right\}_{j=1}^{N}. \tag{S.25}$$

Let $\bar{\xi}_{\cdot k} = n^{-1}\sum_{t=1}^{n}\xi_{tk}$ and $\bar{\xi}_{\cdot k,h} = n^{-1}\sum_{t=1}^{n-h}\xi_{t+h,k}$, then

$$n^{-1}\sum_{t=1}^{n-h}\chi_{t+h}(x')\frac{1}{n}\sum_{i'=1}^{n}\tilde{\chi}_{i'}(x) = \mathbf{B}(x)^{\top}\mathbf{V}_{n,p}^{-1}\left\{\sum_{k=1}^{\infty}\phi_k(x')\,\bar{\xi}_{\cdot k,h}\sum_{k'=1}^{\infty}\langle B_{\ell,p},\phi_{k'}\rangle_N\,\bar{\xi}_{\cdot k'}\right\}_{\ell=1}^{J_s+p}$$
$$= \mathbf{B}(x)^{\top}\mathbf{V}_{n,p}^{-1}\left\{\sum_{k=1}^{\infty}\phi_k(x')\,\bar{\xi}_{\cdot k,h}\sum_{k'=1}^{\infty}\langle B_{\ell,p},\phi_{k'}\rangle_N\,\bar{\xi}_{\cdot k'}\right\}_{\ell=1}^{J_s+p}.$$

Now uniformly for $1 \le \ell \le J_s + p, x \in [0,1]$, one has

$$\left|\sum_{k=1}^{\infty}\phi_k(x')\,\bar{\xi}_{\cdot k,h}\sum_{k'=1}^{\infty}\langle B_{\ell,p},\phi_{k'}\rangle_N\,\bar{\xi}_{\cdot k'}\right| \le CJ_s^{-1}\mathrm{S},$$

where $\mathrm{S} = \sum_{k=1}^{\infty}\|\phi_k\|_\infty\sum_{k'=1}^{\infty}\|\phi_{k'}\|_\infty\left|\bar{\xi}_{\cdot k,h}\bar{\xi}_{\cdot k'}\right|$. In addition, one has

$$\mathrm{E}\left|\bar{\xi}_{\cdot k,h}\bar{\xi}_{\cdot k'}\right| \le \left(\mathrm{E}\left|\bar{\xi}_{\cdot k,h}\right|^2\right)^{1/2}\left(\mathrm{E}\left|\bar{\xi}_{\cdot k}\right|^2\right)^{1/2}$$

and

$$\begin{aligned}
\mathrm{E}\left|\bar{\xi}_{\cdot k}\right|^2 &= \mathrm{E}\left(n^{-1}\sum_{t=1}^{n}\xi_{tk}\right)^2 = n^{-1} + 2n^{-2}\sum_{1\le t<t'\le n}\mathrm{E}\xi_{tk}\xi_{t'k} \\
&= n^{-1} + 2n^{-2}\sum_{1\le t<t'\le n}\gamma_k(|t-t'|) = n^{-1} + 2n^{-2}\sum_{l=1}^{n-1}(n-l)\gamma_k(l) \\
&\le n^{-1} + 2n^{-1}\sum_{l=1}^{\infty}|\gamma_k(l)| \le cn^{-1}.
\end{aligned}$$

Similarly, one has $\mathrm{E}\left|\bar{\xi}_{\cdot k,h}\right|^2 \leq cn^{-1}$, and

$$\begin{aligned}
\mathrm{E}\,\mathrm{S} &\leq \sum_{k=1}^{\infty}\|\phi_k\|_\infty \sum_{k'=1}^{\infty}\|\phi_{k'}\|_\infty \max_{1\leq k,k'\leq\infty} \mathrm{E}\left|\bar{\xi}_{\cdot k,h}\bar{\xi}_{\cdot k'}\right| \\
&\leq \sum_{k=1}^{\infty}\|\phi_k\|_\infty \sum_{k'=1}^{\infty}\|\phi_{k'}\|_\infty \max_{1\leq k,k'\leq\infty} \left(\mathrm{E}\left|\bar{\xi}_{\cdot k,h}\right|^2\right)^{1/2}\left(\mathrm{E}\left|\bar{\xi}_{\cdot k}\right|^2\right)^{1/2} \\
&\leq cn^{-1}\sum_{k=1}^{\infty}\|\phi_k\|_\infty \sum_{k'=1}^{\infty}\|\phi_{k'}\|_\infty .
\end{aligned}$$

As Assumption (A4) guarantees that $\sum_{k=1}^{\infty}\|\phi_k\|_\infty < \infty$, while $\left\|\mathbf{V}_{n,p}^{-1}\right\| \leq CJ_s$ for large N, one obtains that

$$\sup_{x,x'\in[0,1]}\left|\mathbf{B}(x)^\top\mathbf{V}_{n,p}^{-1}\left\{\sum_{k=1}^{\infty}\phi_k\left(x'\right)\bar{\xi}_{\cdot k,h}\sum_{k'=1}^{\infty}\left\langle B_{\ell,p},\phi_{k'}\right\rangle_N\bar{\xi}_{\cdot k'}\right\}_{\ell=1}^{J_s+p}\right| = \mathcal{O}_p\left(n^{-1/2}\right). \tag{S.26}$$

The lemma holds.

0.3 Proofs of the Propositions and Theorem 1

Proof of Proposition 1

Since $C_h(x,x') \equiv 0$ for all $x,x' \in [0,1]$ and $h > q$, according to (2.1), one obtains that $\gamma_k(h) = 0$ for all $k \in \mathbb{N}_+, h > q$. Proposition 3.2.1 of Brockwell and Davis (1991) then entails that for any $k \in \mathbb{N}_+, \{\xi_{tk}\}_{t=-\infty}^{\infty}$ is MA(q_k) where $q_k \leq q$. On the other hand, $C_q(x,x') \neq 0$ for $q \in \mathbb{N}_+$ so there exists $k_0 \in \mathbb{N}_+$ such that $\gamma_{k_0}(q) \neq 0$, hence $q_{k_0} = q$, so $\left\{\xi_{tk_0}\right\}_{t=-\infty}^{\infty}$ is MA(q). Equation (3.2.3) of Brockwell and Davis (1991) ensures that the white noise innovations in MA(q_k) expression of $\{\xi_{tk}\}_{t=-\infty}^{\infty}$ is unique up to a factor of ±1, while invertibility of MA(∞) in (1.4) implies the following identity of linear subspaces

$$\overline{\mathrm{sp}}\left\{\xi_{sk}, -\infty < s \leq t\right\} \equiv \overline{\mathrm{sp}}\left\{\zeta_{sk}, -\infty < s \leq t\right\}, k \in \mathbb{N}_+.$$

Consequently the strong white noise series $\{\zeta_{tk}\}_{t=-\infty}^{\infty}$ equals the white noise innovations of MA(q_k) for $\{\xi_{tk}\}_{t=-\infty}^{\infty}$ up to factors of ± 1. Thus

$$\xi_{tk} = \sum_{t'=0}^{q} a_{t',k}\zeta_{t-t',k}, \ \ a.s. \ , t \in \mathbb{Z}, k \in \mathbb{N}_+.$$

Hence $\{\chi_t(\cdot)\}_{t=-\infty}^{\infty}$ is an FMA(q) series with $\chi_t(\cdot) = \sum_{t'=0}^{q} A_{t'}\zeta_{t-t'}(\cdot)$.

Proof of Proposition 2

We decompose the difference between $\hat{C}_h(x,x')$ and $\tilde{C}_h(x,x')$ into the following three terms:

$$\hat{C}_h(x,x') - \tilde{C}_h(x,x') = \mathrm{I}_h(x,x') + \mathrm{II}_h(x,x') + \mathrm{III}_h(x,x'),$$

where

$$\mathrm{I}_h(x,x') = n^{-1}\sum_{t=1}^{n-h}\{\hat{\chi}_t(x) - \chi_t(x)\}\left\{\hat{\chi}_{t+h}(x') - \chi_{t+h}(x')\right\},$$
$$\mathrm{II}_h(x,x') = n^{-1}\sum_{t=1}^{n-h}\chi_{t+h}(x')\{\hat{\chi}_t(x) - \chi_t(x)\},$$
$$\mathrm{III}_h(x,x') = n^{-1}\sum_{t=1}^{n-h}\chi_t(x)\{\chi_t(x') - \chi_t(x')\}.$$

Note that by (S.5), $\sup_{(x,x)\in[0,1]^2}|\mathrm{I}_h(x,x')| \le \max_{1\le t\le n}\|\hat{\chi}_t - \chi_t\|_{\infty}^2$. According to (S.21), one can obtain

$$\max_{1\le t\le n}\|\hat{\chi}_t - \chi_t\|_{\infty}^2 = \mathcal{O}_p\left\{J_s^{-2p^*}(n\log n)^{4/r_0} + J_s N^{-1}\log N + J_s^2 N^{2(\beta_2-1)} + n^{-1}\right\} = \scriptstyle{\mathcal{O}}\left(n^{-1/2}\right),$$

By (S.5), one has

$$\begin{aligned}\mathrm{II}_h(x,x') =& n^{-1}\sum_{t=1}^{n-h}\chi_{t+h}(x')\{\hat{\chi}_t(x) - \chi_t(x)\}\\ =& n^{-1}\left[\sum_{t=1}^{n-h}\chi_{t+h}(x')\{\tilde{\chi}_t(x) - \chi_t(x)\} + \sum_{t=1}^{n-h}\chi_{t+h}(x')\tilde{\varepsilon}_t(x)\right]\\ &- n^{-2}\left[\sum_{t=1}^{n-h}\chi_{t+h}(x')\sum_{i'=1}^{n}\tilde{\chi}_{i'}(x)\,dx + \sum_{t=1}^{n-h}\chi_{t+h}(x')\sum_{i'=1}^{n}\tilde{\varepsilon}_{i'}(x)\right].\end{aligned}$$

Similar to the proof of Lemma S.13, it is easy to see

$$\sup_{x,x'\in[0,1]} n^{-2}\left|\sum_{t=1}^{n-h}\chi_{t+h}(x')\sum_{i'=1}^{n}\tilde{\varepsilon}_{i'}(x)\right| = \mathcal{O}_p\left(n^{-1/2}\right).$$

Consequently, by Lemmas S.12, S.13 and S.14, one has

$$\sup_{x,x'\in[0,1]}\left|\mathrm{II}_h(x,x')\right| = \sup_{x,x'\in[0,1]} n^{-1}\left|\sum_{t=1}^{n-h}\chi_{t+h}(x')\left\{\hat{\chi}_t(x)-\chi_t(x)\right\}\right| = \mathcal{O}_p\left(n^{-1/2}\right).$$

Similarly, one can show that $\sup_{x,x'\in[0,1]}\left|\mathrm{III}_h(x,x')\right| = \sup_{x,x'\in[0,1]}\left|\mathrm{II}_h(x,x')\right|$. Consequently,

$$\sup_{x,x'\in[0,1]}\left|\hat{C}_h(x,x')-\tilde{C}_h(x,x')\right| = \sup_{x,x'\in[0,1]}\left|\mathrm{I}_h(x,x')+\mathrm{II}_h(x,x')+\mathrm{III}_h(x,x')\right| = \mathcal{O}_p\left(n^{-1/2}\right).$$

Proof of Theorem 1

Without loss of generality, considering $h\in\mathbb{N}$, denote $\bar{\xi}_{\cdot kk',h} = n^{-1}\sum_{t=1}^{n-h}\xi_{tk}\xi_{t+h,k'}$ and $\mathcal{F}_{m,h} = \sigma\left(\bar{\xi}_{\cdot 11,h},\bar{\xi}_{\cdot 12,h},\ldots,\bar{\xi}_{\cdot 1m,h},\bar{\xi}_{\cdot 21,h},\right.$ $\left.\ldots,\bar{\xi}_{\cdot m-1,m,h},\ldots,\bar{\xi}_{\cdot m1,h},\ldots,\bar{\xi}_{\cdot mm,h}\right)$, so that $\mathcal{F}_{2,h}\subseteq\mathcal{F}_{3,h}\subseteq\mathcal{F}_{4,h}\subseteq\cdots$ is an increasing sequence of σ-fields.

Denote

$$\begin{aligned} S_m(x,x') =& n^{1/2}\sum_{1\le k\ne k'\le m}\bar{\xi}_{\cdot kk',h}\phi_k(x)\phi_{k'}(x) \\ &+ n^{1/2}\sum_{1\le k\le m}\left\{\bar{\xi}_{\cdot kk,h}-\frac{n-h}{n}\gamma_k(h)\right\}\phi_k(x)\phi_k(x'), \end{aligned}$$

for $m\in\{1,\ldots,k_n\}$, where k_n satisfies Assumption (A4). We show that $S_m(x,x')$ is a martingale process in $x,x'\in[0,1]$.

Define $D_m(x,x') = S_m(x,x') - S_{m-1}(x,x')$, thus,

$$\begin{aligned} D_m(x,x') \;=\; & n^{1/2}\left[\sum_{k=1}^{m-1}\left\{\bar{\xi}_{\cdot km,h}\phi_k(x)\phi_m(x')+\bar{\xi}_{\cdot mk,h}\phi_m(x)\phi_k(x')\right\}\right. \\ &\left.+\left\{\bar{\xi}_{\cdot mm,h}-\frac{n-h}{n}\gamma_m(h)\right\}\phi_m(x)\phi_m(x')\right], \end{aligned}$$

which is $\mathcal{F}_{m,h}$-measurable. While notice that for any t,

$$\begin{aligned}
&\mathrm{E}\left\{ D_m(x,x') \middle| \mathcal{F}_{m-1,h}\right\} \\
&= n^{1/2}\mathrm{E}\left[\sum_{k=1}^{m-1}\left\{\bar{\xi}_{\cdot km,h}\phi_k(x)\phi_m(x') + \bar{\xi}_{\cdot mk,h}\phi_m(x)\phi_k(x')\right\}\right. \\
&\left. + \left\{\bar{\xi}_{\cdot mm,h} - \frac{n-h}{n}\gamma_m(h)\right\}\phi_m(x)\phi_m(x') \middle| \mathcal{F}_{m-1,h}\right] \\
&= n^{-1/2}\left[\mathrm{E}\left\{\sum_{t=1}^{n-h}\sum_{k=1}^{m-1}\left\{\xi_{tm}\xi_{t+h,k}\phi_m(x)\phi_k(x') + \xi_{t+h,m}\xi_{tk}\phi_k(x)\phi_m(x')\right\} \middle| \mathcal{F}_{m-1,h}\right\}\right] \\
&\quad + n^{1/2}\mathrm{E}\left\{\left\{\bar{\xi}_{\cdot mm,h} - \frac{n-h}{n}\gamma_m(h)\right\}\phi_m(x)\phi_m(x') \middle| \mathcal{F}_{m-1,h}\right\} = 0,
\end{aligned}$$

which implies that $\{D_m(x,x'), m = 2, 3, \ldots\}$ is a martingale difference process with respect to $\{\mathcal{F}_{m-1,h}, m = 2, 3, \ldots\}$.

Next denote

$$\mathrm{E}\left\{ D_m^2(x,x') \middle| \mathcal{F}_{m-1,h}\right\} = \mathrm{V}_m^{(1)}(x,x') + \mathrm{V}_m^{(2)}(x,x') + \mathrm{V}_m^{(3)}(x,x'), \quad \text{(S.27)}$$

in which

$$\begin{aligned}
\mathrm{V}_m^{(1)}(x,x') =& n^{-1}\mathrm{E}\left[\left\{\sum_{t=1}^{n-h}\sum_{k=1}^{m-1}\left\{\xi_{tm}\xi_{t+h,k}\phi_m(x)\phi_k(x') + \xi_{t+h,m}\xi_{tk}\phi_k(x)\phi_m(x')\right\}\right\}^2 \middle| \mathcal{F}_{m-1,h}\right], \\
\mathrm{V}_m^{(2)}(x,x') =& n\mathrm{E}\left[\left\{\bar{\xi}_{\cdot mm,h} - \frac{n-h}{n}\gamma_m(h)\right\}^2\phi_m^2(x)\phi_m^2(x') \middle| \mathcal{F}_{m-1,h}\right], \\
\mathrm{V}_m^{(3)}(x,x') =& 2\mathrm{E}\left[\left\{\sum_{t=1}^{n-h}\sum_{k=1}^{m-1}\left\{\xi_{tm}\xi_{t+h,k}\phi_m(x)\phi_k(x') + \xi_{t+h,m}\xi_{tk}\phi_k(x)\phi_m(x')\right\}\right\}\right. \\
&\left. \times \left\{\bar{\xi}_{\cdot mm,h} - \frac{n-h}{n}\gamma_m(h)\right\}\phi_m(x)\phi_m(x') \middle| \mathcal{F}_{m-1,h}\right].
\end{aligned}$$

Moreover, one can show that

$$\begin{aligned}
\mathrm{V}_m^{(1)}(x,x') &= n^{-1}\mathrm{E}\left[\left\{\sum_{t=1}^{n-h}\sum_{k=1}^{m-1}\left\{\xi_{tm}\xi_{t+h,k}\phi_m(x)\phi_k(x')+\xi_{t+h,m}\xi_{tk}\phi_k(x)\phi_m(x')\right\}\right\}^2\middle|\mathcal{F}_{m-1,h}\right]\\
&= n^{-1}\mathrm{E}\sum_{k=1}^{m-1}\left[\left\{\sum_{t=1}^{n-h}\left\{\xi_{tm}\xi_{t+h,k}\phi_m(x)\phi_k(x')+\xi_{t+h,m}\xi_{tk}\phi_k(x)\phi_m(x')\right\}\right\}^2\middle|\mathcal{F}_{m-1,h}\right]\\
&+ n^{-1}\mathrm{E}\sum_{k\neq k'}^{m-1}\left[\left\{\sum_{t=1}^{n-h}\left\{\xi_{tm}\xi_{t+h,k}\phi_m(x)\phi_k(x')+\xi_{t+h,m}\xi_{tk}\phi_k(x)\phi_m(x')\right\}\right\}\right.\\
&\left.\times\left\{\sum_{t=1}^{n-h}\left\{\xi_{tm}\xi_{t+h,k'}\phi_m(x)\phi_{k'}(x')+\xi_{t+h,m}\xi_{tk'}\phi_{k'}(x)\phi_m(x')\right\}\right\}\middle|\mathcal{F}_{m-1,h}\right].
\end{aligned}$$

Note that

$$\begin{aligned}
& n^{-1}\mathrm{E}\sum_{k=1}^{m-1}\left[\left\{\sum_{t=1}^{n-h}\left\{\xi_{tm}\xi_{t+h,k}\phi_m(x)\phi_k(x')+\xi_{t+h,m}\xi_{tk}\phi_k(x)\phi_m(x')\right\}\right\}^2\middle|\mathcal{F}_{m-1,h}\right]\\
=\ & n^{-1}\sum_{k=1}^{m-1}\sum_{t,t'=1}^{n-h}\left[\gamma_m(|t-t'|)\left\{\xi_{t+h,k}\xi_{t'+h,k}\phi_m^2(x)\phi_k^2(x')+\xi_{tk}\xi_{t'k}\phi_k^2(x)\phi_m^2(x')\right\}\right.\\
& \left.+\left\{\gamma_m(|t'-t+h|)\xi_{t+h,k}\xi_{t'k}\ +\ \gamma_m(|t-t'+h|)\xi_{t'+h,k}\xi_{tk}\right\}\phi_m(x)\phi_k(x)\phi_m(x')\phi_k(x')\right],
\end{aligned}$$

therefore, one has when $n\to\infty$,

$$\sum_{m=2}^{k_n}\mathrm{V}_m^{(1)}(x,x')\to\sum_{k\neq k'}^{\infty}\left\{2\sum_{l=1}^{\infty}\gamma_k(l)\gamma_{k'}(l)+1\right\}\phi_k^2(x)\phi_{k'}^2(x')$$

$$+\sum_{k\neq k'}^{\infty}\left\{\gamma_k(h)\gamma_{k'}(h)+2\sum_{l=1}^{\infty}\gamma_k(l+h)\gamma_{k'}(|l-h|)\right\}\phi_k(x)\phi_{k'}(x)\phi_k(x')\phi_{k'}(x')<\infty$$

Note that

$$\begin{aligned}
\mathrm{V}_m^{(2)}(x,x') &= n\mathrm{E}\left[\left\{\bar{\xi}_{\cdot mm,h}-\frac{n-h}{n}\gamma_m(h)\right\}^2\phi_m^2(x)\phi_m^2(x')\right] \\
&= \left\{n^{-1}\mathrm{E}\left(\sum_{t=1}^{n-h}\xi_{tm}\xi_{t+h,m}\right)^2-\frac{(n-h)^2}{n}\gamma_m^2(h)\right\}\phi_m^2(x)\phi_m^2(x') \\
&= \left\{n^{-1}\sum_{t'\neq t}\mathrm{E}\xi_{tm}\xi_{t+h,m}\xi_{t'm}\xi_{t'+h,m}-\frac{(n-h)^2}{n}\gamma_m^2(h)\right\}\phi_m^2(x)\phi_m^2(x') \\
&\quad +\frac{n-h}{n}\mathrm{E}\xi_{1m}^2\xi_{1+h,m}^2\phi_m^2(x)\phi_m^2(x').
\end{aligned}$$

Now one considers that

$$\begin{aligned}
& n^{-1}\sum_{t'\neq t}\mathrm{E}\xi_{tm}\xi_{t+h,m}\xi_{t'm}\xi_{t'+h,m}-\frac{(n-h)^2}{n}\gamma_m^2(h) \\
=\;& 2n^{-1}\sum_{l=1}^{n-h-1}(n-h-l)\,\mathrm{E}\xi_{1m}\xi_{1+h,m}\xi_{1+l,m}\xi_{1+l+h,m}-\frac{(n-h)^2}{n}\gamma_m^2(h) \\
=\;& 2n^{-1}\sum_{l=1}^{n-h-1}(n-h-l)\sum_{0\le t,t_1,t_2,t_3\le\infty}a_{tm}a_{t_1m}a_{t_2m}a_{t_3m}\mathrm{E}\zeta_{1-t,m}\zeta_{1+h-t_1,m}\zeta_{1+l-t_2,m}\zeta_{1+l+h-t_3,m} \\
& -\frac{(n-h)^2}{n}\gamma_m^2(h) \\
=\;& 2n^{-1}\sum_{l=1}^{n-h-1}(n-h-l)\,\mathrm{E}\left(\zeta_{0m}^4-3\right)\sum_{t=0}^{\infty}a_{tm}a_{t+h,m}a_{t+l,m}a_{t+l+h,m} \\
& +2n^{-1}\sum_{l=1}^{n-h-1}(n-h-l)\left\{\gamma_m^2(l)+\gamma_m(l+h)\gamma_m(|l-h|)\right\} \\
& +2n^{-1}\sum_{l=1}^{n-h-1}(n-h-l)\gamma_m^2(h)-\frac{(n-h)^2}{n}\gamma_m^2(h) \\
=\;& 2n^{-1}\sum_{l=1}^{n-h-1}(n-h-l)\,\mathrm{E}\left(\zeta_{0m}^4-3\right)\sum_{t=0}^{\infty}a_{tm}a_{t+h,m}a_{t+l,m}a_{t+l+h,m} \\
& +2n^{-1}\sum_{l=1}^{n-h-1}(n-h-l)\left\{\gamma_m^2(l)+\gamma_m(l+h)\gamma_m(|l-h|)\right\}-\frac{n-h}{n}\gamma_m^2(h),
\end{aligned}$$

so one has that

$$\begin{aligned}\sum_{m=2}^{k_n} \mathrm{V}_m^{(2)}(x,x') &\to 2\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\left(\mathrm{E}\zeta_{0m}^4-3\right)\sum_{t=0}^{\infty} a_{tm}a_{t+h,m}a_{t+l,m}a_{t+l+h,m}\phi_m^2(x)\,\phi_m^2(x') \\ &\quad +2\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\left\{\gamma_m^2(l)+\gamma_m(l+h)\,\gamma_m(|l-h|)\right\}\phi_m^2(x)\,\phi_m^2(x') \\ &\quad +\sum_{m=1}^{\infty}\left\{1+\left(\mathrm{E}\zeta_{0m}^4-3\right)\sum_{t=0}^{\infty} a_{tm}^2a_{t+h,m}^2+\gamma_m^2(h)\right\}\phi_m^2(x)\,\phi_m^2(x') \\ &< \infty.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathrm{V}_m^{(3)}(x,x') &= 2\sum_{t=1}^{n-h}\mathrm{E}\left[\xi_{tm}\left\{\bar{\xi}_{\cdot mm,h}-\frac{n-h}{n}\gamma_m(h)\right\}\right]\sum_{k=1}^{m-1}\xi_{t+h,k}\phi_m^2(x)\,\phi_m(x')\,\phi_k(x') \\ &\quad +2\sum_{t=1}^{n-h}\mathrm{E}\left[\xi_{t+h,m}\left\{\bar{\xi}_{\cdot mm,h}-\frac{n-h}{n}\gamma_m(h)\right\}\right]\sum_{k=1}^{m-1}\xi_{tk}\phi_m(x)\,\phi_m^2(x')\,\phi_k(x').\end{aligned}$$

Thus

$$\begin{aligned}\mathrm{V}_m^{(3)}(x,x') &= 2n^{-1}\sum_{t=1}^{n-h}\mathrm{E}\left(\xi_{tm}\sum_{t'=1}^{n-h}\xi_{t'm}\xi_{t'+h,m}\right)\sum_{k=1}^{m-1}\xi_{t+h,k}\phi_m^2(x)\,\phi_m(x')\,\phi_k(x') \\ &\quad +2n^{-1}\sum_{t=1}^{n-h}\mathrm{E}\left(\xi_{t+h,m}\sum_{t'=1}^{n-h}\xi_{t'm}\xi_{t'+h,m}\right)\sum_{k=1}^{m-1}\xi_{tk}\phi_m(x)\,\phi_m^2(x')\,\phi_k(x').\end{aligned}$$

Next, notice that

$$\sup_{x,x'\in[0,1]}\left|\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\mathrm{E}\left(\xi_{1m}\sum_{t'=1}^{\infty}\xi_{t'm}\xi_{t'+h,m}\right)\phi_m^2(x)\,\phi_m(x')\,\phi_k(x')\right|<\infty,$$

$$\sup_{x,x'\in[0,1]}\left|\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\mathrm{E}\left(\xi_{1+h,m}\sum_{t'=1}^{\infty}\xi_{t'm}\xi_{t'+h,m}\right)\phi_m(x)\,\phi_m^2(x')\,\phi_k(x')\right|<\infty.$$

Therefore, one has

$$\begin{aligned}\sum_{m=2}^{k_n}\mathrm{V}_m^{(3)}(x,x') &\to \sum_{k<m}\mathrm{E}\left(\xi_{1m}\sum_{t'=1}^{n-h}\xi_{t'm}\xi_{t'+h,m}\right)\left(2n^{-1}\sum_{t=1}^{n-h}\xi_{t+h,k}\right)\phi_m^2(x)\,\phi_m(x')\,\phi_k(x') \\ &\quad +\sum_{k<m}\mathrm{E}\left(\xi_{1+h,m}\sum_{t'=1}^{n-h}\xi_{t'm}\xi_{t'+h,m}\right)\left(2n^{-1}\sum_{t=1}^{n-h}\xi_{tk}\right)\phi_m(x)\,\phi_m^2(x')\,\phi_k(x')\to_p 0,\end{aligned}$$

as $n \to \infty$.

According to (S.27), as $n \to \infty$, one has

$$\sum_{m=2}^{k_n} \mathrm{E}\left\{ D_m^2(x,x') \middle| \mathcal{F}_{m-1,h} \right\} \to_p \sum_{k \neq k'}^{\infty} \left\{ 2\sum_{l=1}^{\infty} \gamma_k(l)\gamma_{k'}(l) + 1 \right\} \phi_k^2(x)\phi_{k'}^2(x')$$
$$+ \sum_{k \neq k'}^{\infty} \left\{ \gamma_k(h)\gamma_{k'}(h) + 2\sum_{l=1}^{\infty} \gamma_k(l+h)\gamma_{k'}(|l-h|) \right\} \phi_k(x)\phi_{k'}(x)\phi_k(x')\phi_{k'}(x')$$
$$+ 2\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} \left(\mathrm{E}\zeta_{0m}^4 - 3\right) \sum_{t=0}^{\infty} a_{tm}a_{t+h,m}a_{t+l,m}a_{t+l+h,m}\phi_m^2(x)\phi_m^2(x')$$
$$+ 2\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} \left\{ \gamma_m^2(l) + \gamma_m(l+h)\gamma_m(|l-h|) \right\} \phi_m^2(x)\phi_m^2(x')$$
$$+ \sum_{m=1}^{\infty} \left\{ 1 + \left(\mathrm{E}\zeta_{0m}^4 - 3\right)\sum_{t=0}^{\infty} a_{tm}^2 a_{t+h,m}^2 + \gamma_m^2(h) \right\} \phi_m^2(x)\phi_m^2(x')$$

$$= \sum_{k \neq k'}^{\infty}\sum_{l=-\infty}^{\infty} \gamma_k(l)\gamma_{k'}(l)\phi_k^2(x)\phi_{k'}^2(x')$$
$$+ \sum_{k \neq k'}^{\infty}\sum_{l=-\infty}^{\infty} \gamma_k(l+h)\gamma_{k'}(l-h)\phi_k(x)\phi_{k'}(x)\phi_k(x')\phi_{k'}(x')$$
$$+ \sum_{m=1}^{\infty} \left(\mathrm{E}\zeta_{0m}^4 - 3\right)\gamma_m^2(h)\phi_m^2(x)\phi_m^2(x')$$
$$+ \sum_{m=1}^{\infty}\sum_{l=-\infty}^{\infty} \left\{ \gamma_m^2(l) + \gamma_m(l+h)\gamma_m(l-h) \right\} \phi_m^2(x)\phi_m^2(x').$$

$$= \sum_{l=-\infty}^{\infty} \left\{ \sum_{k=1}^{\infty} \gamma_k(l)\phi_k^2(x) \right\} \left\{ \sum_{k=1}^{\infty} \gamma_k(l)\phi_k^2(x') \right\}$$
$$+ \sum_{l=-\infty}^{\infty} \left\{ \sum_{k=1}^{\infty} \gamma_k(l+h)\phi_k(x)\phi_k(x') \right\} \left\{ \sum_{k=1}^{\infty} \gamma_k(l-h)\phi_k(x)\phi_k(x') \right\}$$
$$+ \sum_{m=1}^{\infty} \left(\mathrm{E}\zeta_{0m}^4 - 3\right)\gamma_m^2(h)\phi_m^2(x)\phi_m^2(x')$$

$$= \sum_{l=-\infty}^{\infty} C_l(x,x)\, C_l(x',x') + \sum_{l=-\infty}^{\infty} C_{l+h}(x,x')\, C_{l-h}(x,x') + \sum_{m=1}^{\infty} \left(\mathrm{E}\zeta_{0m}^4 - 3\right) \gamma_m^2(h)\, \phi_m^2(x)\, \phi_m^2(x').$$

Denote by $\mathrm{E}\left\{ D_m^3(x,x') \middle| \mathcal{F}_{m-1,h} \right\} = d_m^{(1)}(x,x') + 3d_m^{(2)}(x,x') + 3d_m^{(3)}(x,x') + d_m^{(4)}(x,x')$, where

$$d_m^{(1)}(x,x') = n^{-3/2}\mathrm{E}\left[\left\{\sum_{t=1}^{n-h}\sum_{k=1}^{m-1}\left\{\xi_{tm}\xi_{t+h,k}\phi_m(x)\,\phi_k(x') + \xi_{t+h,m}\xi_{tk}\phi_k(x)\,\phi_m(x')\right\}\right\}^3 \middle| \mathcal{F}_{m-1,h}\right],$$

$$d_m^{(2)}(x,x') = n^{-1/2}\mathrm{E}\left[\left\{\sum_{t=1}^{n-h}\sum_{k=1}^{m-1}\left\{\xi_{tm}\xi_{t+h,k}\phi_m(x)\,\phi_k(x') + \xi_{t+h,m}\xi_{tk}\phi_k(x)\,\phi_m(x')\right\}\right\}^2 \times \left\{\bar{\xi}_{\cdot mm,h} - \frac{n-h}{n}\gamma_m(h)\right\}\phi_m(x)\,\phi_m(x') \middle| \mathcal{F}_{m-1,h}\right],$$

$$d_m^{(3)}(x,x') = n^{1/2}\mathrm{E}\left[\left\{\sum_{t=1}^{n-h}\sum_{k=1}^{m-1}\left\{\xi_{tm}\xi_{t+h,k}\phi_m(x)\,\phi_k(x') + \xi_{t+h,m}\xi_{tk}\phi_k(x)\,\phi_m(x')\right\}\right\} \times \left\{\bar{\xi}_{\cdot mm,h} - \frac{n-h}{n}\gamma_m(h)\right\}^2\phi_m^2(x)\,\phi_m^2(x') \middle| \mathcal{F}_{m-1,h}\right],$$

$$d_m^{(4)}(x,x') = n^{3/2}\mathrm{E}\left[\left\{\bar{\xi}_{\cdot mm,h} - \frac{n-h}{n}\gamma_m(h)\right\}^3\phi_m^3(x)\,\phi_m^3(x') \middle| \mathcal{F}_{m-1,h}\right].$$

Applying similar arguments as Lemma 6 of Cao et al. (2016), $\sum_{m=2}^{k_n}\mathrm{E}\left\{ d_m^{(i)}(x,x') \middle| \mathcal{F}_{m-1,h}\right\} \to_p 0$, for $i = 1,2,3,4$. Hence $\forall\epsilon > 0$, $\sup_{x,x'\in[0,1]}\sum_{m=2}^{k_n}\mathrm{E}\left\{ D_m^3(x,x')\, I\left(D_m^2(x,x') > \epsilon\right) \middle| \mathcal{F}_{m-1,h}\right\} \to_p 0$.

By the uniform central limit theorem, one has $\sqrt{n}\Delta_h(x,x') = S_m(x,x') \to_D \varphi_h(x,x')$, as $n\to\infty$, where $\varphi_h(x,x')$ is a Gaussian random field such that

$\mathrm{E}\varphi_h(x,x') \equiv 0$ with variance function

$$\begin{aligned}\Xi_h(x,x') &= \sum_{k\neq k'}^{\infty}\sum_{l=-\infty}^{\infty}\gamma_k(l)\gamma_{k'}(l)\phi_k^2(x)\phi_{k'}^2(x') \\ &+\sum_{k\neq k'}^{\infty}\sum_{l=-\infty}^{\infty}\gamma_k(l+h)\gamma_{k'}(l-h)\phi_k(x)\phi_{k'}(x)\phi_k(x')\phi_{k'}(x') \\ &+\sum_{m=1}^{\infty}\left(\mathrm{E}\zeta_{0m}^4-3\right)\gamma_m^2(h)\phi_m^2(x)\phi_m^2(x') \\ &+\sum_{m=1}^{\infty}\sum_{l=-\infty}^{\infty}\left\{\gamma_m^2(l)+\gamma_m(l+h)\gamma_m(l-h)\right\}\phi_m^2(x)\phi_m^2(x')\end{aligned}$$

$$=\sum_{l=-\infty}^{\infty}\{C_l(x,x)C_l(x',x')+C_{l+h}(x,x')C_{l-h}(x,x')\}+\sum_{m=1}^{\infty}\left(\mathrm{E}\zeta_{0m}^4-3\right)\gamma_m^2(h)\phi_m^2(x)\phi_m^2(x').$$

and covariance function

$$\begin{aligned}\Omega_h(x,x',y,y') &= \mathrm{Cov}\left\{\varphi_h(x,x'),\varphi_h(y,y')\right\} \\ &= \sum_{k\neq k'}^{\infty}\sum_{l=-\infty}^{\infty}\gamma_k(l)\gamma_{k'}(l)\phi_k(x)\phi_{k'}(x')\phi_k(y)\phi_{k'}(y')\end{aligned}$$

$$\begin{aligned}&+\sum_{k\neq k'}^{\infty}\sum_{l=-\infty}^{\infty}\gamma_k(l+h)\gamma_{k'}(l-h)\phi_k(x)\phi_{k'}(x')\phi_k(y)\phi_{k'}(y') \\ &+\sum_{m=1}^{\infty}\left(\mathrm{E}\zeta_{0m}^4-3\right)\gamma_m^2(h)\phi_m(x)\phi_m(x')\phi_m(y)\phi_m(y') \\ &+\sum_{m=1}^{\infty}\sum_{l=-\infty}^{\infty}\left\{\gamma_m^2(l)+\gamma_m(l+h)\gamma_m(l-h)\right\}\phi_m(x)\phi_m(x')\phi_m(y)\phi_m(y')\end{aligned}$$

$$\begin{aligned}&= \sum_{l=-\infty}^{\infty}C_l(x,y)C_l(x',y')+\sum_{l=-\infty}^{\infty}C_{l+h}(x,x')C_{l-h}(y,y') \\ &\quad+\sum_{m=1}^{\infty}\left(\mathrm{E}\zeta_{0m}^4-3\right)\gamma_m^2(h)\phi_m(x)\phi_m(x')\phi_m(y)\phi_m(y')\end{aligned}$$

for $x,x',y,y'\in[0,1]$. The theorem is proved.

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