Supplementary material for "Inference for dependent error functional data with application to event related potentials "

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This supplement contains all the proofs for the main results.

S.1 Preliminaries

Throughout this section, $\mathcal{O}_p$ (or $\scriptstyle\mathcal{O}_p$) denotes a sequence of random variables of certain order in probability. For instance, $o_p(n^{-1/2})$ means a smaller order than $n^{-1/2}$ in probability, and by $\mathcal{O}_{a.s.}$ (or $o_{a.s.}$) almost surely $\mathcal{O}$ (or o). In addition, $\mathcal{U}_p$ denotes a sequence of random functions which are $\mathcal{O}_p$ uniformly defined in the domain.

For any vector $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$, denote the norm $\|\mathbf{a}\|_r = (|a_1|^r + \cdots + |a_n|^r)^{1/r}$, $1 \le r < +\infty$, $\|\mathbf{a}\|_\infty = \max(|a_1|, \ldots, |a_n|)$. For any matrix $\mathbf{A} = (a_{ij})_{i=1,j=1}^{m,n}$, denote its L_r norm as $\|\mathbf{A}\|_r = \max_{\mathbf{a}\in\mathbb{R}^n\setminus\{\mathbf{0}\}} \|\mathbf{A}\mathbf{a}\|_r / \|\mathbf{a}\|_r$, for $r < +\infty$ and $\|\mathbf{A}\|_\infty = \max_{1\le i\le m} \sum_{j=1}^n |a_{ij}|$. we will introduce the following lemmas needed in the proofs of the main results.

Lemma S.1 *(Theorem 2.6.7, Csőrgő and Révész (1981)) Suppose that* $\{\xi_i\}_{i\in\mathbb{N}_+}$ *are i.i.d. on a probability space* $(\Omega, \mathcal{A}, \mathbb{P})$, *with* $\mathbb{E}(\xi_1) = 0$, $\mathbb{E}(\xi_1^2) = 1$ *and* $H(x) > 0$ $(x \ge 0)$ *is an increasing continuous function such that* $x^{-2-\gamma}H(x)$ *is increasing for some* $\gamma > 0$ *and* $x^{-1}\log H(x)$ *is decreasing with* $\mathbb{E}H(|\xi_1|) < \infty$. *Then there exist random variables* $\left\{\tilde{\xi}_i\right\}_{i\in\mathbb{N}_+}$ *and a Wiener process* $W(t), t \in [0, +\infty)$ *on a new probability space* $\left(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbb{P}}\right)$ *such that* $\left\{\tilde{\xi}_i\right\}_{i\in\mathbb{N}_+} \overset{D}{=} \{\xi_i\}_{i\in\mathbb{N}_+}$, *and constants* $C_1, C_2, a > 0$ *depending only on the distribution of* ξ_1, *such that* $\tilde{\mathbb{P}}\left\{\max_{1\le m\le n}\left|\tilde{S}_m - W(m)\right| > x_n\right\} \le C_2 n \{H(ax_n)\}^{-1}, n \in \mathbb{N}_+$ *for any* $\{x_n\}_{n=1}^\infty$ *satisfying* $H^{-1}(n) < x_n < C_1 (n\log n)^{1/2}$, *and* $\tilde{S}_m = \sum_{i=1}^m \tilde{\xi}_i$ *for any* $m \in \mathbb{N}_+$.

Lemma S.2 *Let* $W_i \sim N\left(0, \sigma_i^2\right), \sigma_i > 0, i = 1, \ldots, n,$ *then for* $n > 2$, $a > 2$

$$\mathbb{P}\left(\max_{1\leq i\leq n} |W_i/\sigma_i| > a\sqrt{\log n}\right) < \sqrt{\frac{2}{\pi}} n^{1-a^2/2}. \tag{S.1}$$

Hence, $\left(\max_{1\leq i\leq n} |W_i|\right) / \left(\max_{1\leq i\leq n} \sigma_i\right) \leq \max_{1\leq i\leq n} |W_i/\sigma_i| = \mathcal{O}_{a.s.}\left(\sqrt{\log n}\right)$.

Proof. Note that

$$\begin{aligned}\mathbb{P}\left(\max_{1\leq i\leq n} \left|\frac{W_i}{\sigma_i}\right| > a\sqrt{\log n}\right) &\leq \sum_{i=1}^{n} \mathbb{P}\left(\left|\frac{W_i}{\sigma_i}\right| > a\sqrt{\log n}\right) \\ &\leq 2n\left\{1 - \Phi\left(a\sqrt{\log n}\right)\right\} < 2n\frac{\phi\left(a\sqrt{\log n}\right)}{a\sqrt{\log n}} \\ &\leq 2n\phi\left(a\sqrt{\log n}\right) = \sqrt{\frac{2}{\pi}} n^{1-a^2/2},\end{aligned}$$

for $n > 2, a > 2$, which proves (S.1). The lemma follows by applying Borel-Cantelli Lemma with choice of $a > 2$.

Lemma S.3 *The inequalities in (7), (8), (9), (10) are consistent and ensure that there exists* γ *which satisfies (11). Under Assumptions (A3), (A6), as* $n \to \infty$,

$$J_s^{-p^*}\left(n\log n\right)^{2/r_0} = o\left(n^{-1/2}\right), \tag{S.2}$$

$$NJ_s^{-1}N^{-v} = o\left(1\right), \tag{S.3}$$

$$N^{-1/2}J_s^{1/2}\sqrt{\log N} = o\left(N^{-\beta_2/2-\theta/4}\right) = o\left(1\right), \tag{S.4}$$

$$N^{\beta_2-1}J_s = o\left(n^{-1/2}\right). \tag{S.5}$$

Proof. Note first that (7) merely specifies appropriate range for Hölder continuity indices μ and ν. Next, since (8) requires that $\theta < 2\nu, \theta < 2p^*/\left(1+p^*\right)$, it follows that $\nu - \theta/2 > 0, 1 - \theta/2 - \theta/2p^* > 0$ and thus there exists β_2 that satisfies (9).

Since (9) compels $\beta_2 < 1-\theta/2-\theta/2p^*$ hence $1-\beta_2-\theta/2 > \theta/2p^*$, so

$$2p^*\left(1-\beta_2-\theta/2\right)-\theta > 0 \tag{S.6}$$

and r_0 exists.

Now $1-\nu < 1-\beta_2-\theta/2$ follows from $\beta_2 < \nu-\theta/2$ in (9), and $\theta\left(1+4/r_0\right)/2p^* < 1-\beta_2-\theta/2$ follows from (10) and (S.6), thus

$$\max\left\{1-\nu, \theta\left(1+4/r_0\right)/2p^*\right\} < 1-\beta_2-\theta/2,$$

so there exists γ that satisfies (11).

Note next that Assumptions (A3), (A6) imply that

$$J_s^{-p^*}\left(n\log n\right)^{2/r_0}n^{1/2} = \mathcal{O}\left(N^{-p^*\gamma}d_N^{-p^*}N^{\theta/2+2\theta/r_0}\left(\log n\right)^{2/r_0}\right) = \mathcal{O}\left(N^{\theta/2+2\theta/r_0-p^*\gamma}d_N^{-p^*}\left(\log n\right)^{2/r_0}\right),$$

since $\theta/2+2\theta/r_0-p^*\gamma < 0$ according to (11), and $d_N^{-p^*} = \mathcal{O}\left(\log^{\tau p^*} N\right)$, so $J_s^{-p^*}\left(n\log n\right)^{2/r_0}n^{1/2} = o\left(1\right)$, proving (S.2).

Since $J_s = N^\gamma d_N$, one computes

$$\begin{aligned} NJ_s^{-1}N^{-v} &= N^{1-\gamma-\nu}d_N^{-1} = \mathcal{O}\left(N^{1-\gamma-\nu}\log N\right), \\ N^{-1/2}J_s^{1/2}\sqrt{\log N} &= N^{(\gamma-1)/2}d_N^{\gamma/2}\sqrt{\log N} = \mathcal{O}\left(N^{(\gamma-1)/2}\log^{(\gamma+1)/2}N\right), \\ N^{\beta_2-1}J_sn^{1/2} &= \mathcal{O}\left(N^{\beta_2-1+\gamma+\theta/2}d_N\right) = \mathcal{O}\left(N^{\beta_2-1+\gamma+\theta/2}\log N\right), \end{aligned}$$

which are $o\left(1\right)$ according to (11) and Equation (S.3)-(S.5) hold.

In the following, the discrepancy $\widehat{m}\left(\cdot\right)-\overline{m}\left(\cdot\right)$ is decomposed into simpler parts, to provides insights on how the B-spline estimator's bias and variance are reduced to order $o_{a.s}\left(n^{-1/2}\right)$ separately.

For any function ϕ defined on $[0,1]$ denote $\boldsymbol{\phi} = \left\{\phi\left(1/N\right),\ldots,\phi\left(N/N\right)\right\}^\top$, in particular, $\boldsymbol{\eta}_i = \left\{\eta_i\left(1/N\right),\ldots,\eta_i\left(N/N\right)\right\}^\top$, $\mathbf{m} = \left\{m\left(1/N\right),\ldots,m\left(N/N\right)\right\}^\top$, $\mathbf{Z}_i = \left\{Z_i\left(1/N\right),\ldots,Z_i\left(N/N\right)\right\}^\top$. In addition, denote by $\mathbf{Y}_i = \left(Y_{i1},\ldots,Y_{iN}\right)^\top$, $\boldsymbol{\varepsilon}_i = \left(\sigma_i\left(1/N\right)\varepsilon_{i1},\ldots,\sigma_i\left(N/N\right)\varepsilon_{iN}\right)^\top$ the response and error vectors of the i-th subject.

According to model (1) and (2), $\mathbf{Y}_i = \boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i, \boldsymbol{\eta}_i = \mathbf{m} + \mathbf{Z}_i$. The design matrix for B-spline regression is

$$\mathbf{B} = \{\mathbf{B}(1/N), \dots, \mathbf{B}(N/N)\}^\top = \begin{pmatrix} B_{1,p}(1/N) & \cdots & B_{J_s+p,p}(1/N) \\ \vdots & \cdots & \vdots \\ B_{1,p}(N/N) & \cdots & B_{J_s+p,p}(N/N) \end{pmatrix}, \qquad \text{(S.7)}$$

where $\mathbf{B}(x) = \{B_{1,p}(x), \dots, B_{J_s+p,p}(x)\}^T$.

For any two L^2 integrable functions $\phi(\cdot)$ and $\varphi(\cdot)$, define their theoretical and empirical inner products as $\langle \phi, \varphi \rangle = \int_{[0,1]} \phi(x)\varphi(x)\,dx$, and $\langle \phi, \varphi \rangle_N = N^{-1}\sum_{j=1}^N \phi(j/N)\varphi(j/N)$ respectively. Correspondingly, the theoretical and empirical norms are $\|\phi\|_2^2 = \langle \phi, \phi \rangle$ and $\|\phi\|_{2,N}^2 = \langle \phi, \phi \rangle_N$. Denote by $\mathbf{V}_{n,p}$ the empirical inner product matrix of B-spline basis $\{B_{\ell,p}(x)\}_{\ell=1}^{J_s+p}$, i.e.

$$\mathbf{V}_{n,p} = \left\{\langle B_{\ell,p}, B_{\ell',p}\rangle_N\right\}_{\ell,\ell'=1}^{J_s+p} = N^{-1}\mathbf{B}^\top\mathbf{B}. \qquad \text{(S.8)}$$

The spline estimator $\widehat{\eta}_i(\cdot)$ allows representation $\widehat{\eta}_i(\cdot) = \mathbf{B}(\cdot)^\top (\mathbf{B}^\top\mathbf{B})^{-1}\mathbf{B}^\top\mathbf{Y}_i$. The approximation error $\widehat{\eta}_i(\cdot) - \eta_i(\cdot)$ is decomposed as

$$\widehat{\eta}_i(\cdot) - \eta_i(\cdot) = \widetilde{\eta}_i(\cdot) - \eta_i(\cdot) + \widetilde{\varepsilon}_i(\cdot), \qquad \text{(S.9)}$$

where

$$\widetilde{\eta}_i(\cdot) = N^{-1}\mathbf{B}(\cdot)^\top \mathbf{V}_{n,p}^{-1}\mathbf{B}^\top\boldsymbol{\eta}_i = \widetilde{m}(\cdot) + \widetilde{Z}_i(\cdot), \qquad \text{(S.10)}$$
$$\widetilde{m}(\cdot) = N^{-1}\mathbf{B}(\cdot)^\top \mathbf{V}_{n,p}^{-1}\mathbf{B}^\top\mathbf{m},\ \widetilde{Z}_i(\cdot) = N^{-1}\mathbf{B}(\cdot)^\top \mathbf{V}_{n,p}^{-1}\mathbf{B}^\top\mathbf{Z}_i, \qquad \text{(S.11)}$$
$$\widetilde{\varepsilon}_i(\cdot) = N^{-1}\mathbf{B}(\cdot)^\top \mathbf{V}_{n,p}^{-1}\mathbf{B}^\top\boldsymbol{\varepsilon}_i, \qquad \text{(S.12)}$$

Consequently, the approximation error of $\widehat{m}(\cdot) - \overline{m}(\cdot)$ can be decomposed as

$$\begin{aligned}\widehat{m}(\cdot) - \overline{m}(\cdot) &= n^{-1}\sum_{i=1}^n \{\widehat{\eta}_i(\cdot) - \eta_i(\cdot)\} \\ &= n^{-1}\sum_{i=1}^n \{\widetilde{\eta}_i(\cdot) - \eta_i(\cdot)\} + n^{-1}\sum_{i=1}^n \widetilde{\varepsilon}_i(\cdot). \end{aligned} \qquad \text{(S.13)}$$

Lemma S.4 *Under Assumptions (A1)–(A6), as $n \to \infty$, one has*

$$\max_{1\le i\le n} \|\widetilde{\eta}_i - \eta_i\|_\infty = \mathcal{O}_{a.s.}\left\{ J_s^{-p^*} (n\log n)^{2/r_0} \right\} = o_{a.s.}\left(n^{-1/2}\right).$$

PROOF The infeasible trajectory $\eta_i(x)$ is written as $\eta_i(x) = m(x) + \sum_{k=1}^\infty \xi_{ik}\phi_k(x)$. For $k = 1, 2, \ldots$, denote $\boldsymbol{\phi}_k = (\phi_k(1/N), \ldots, \phi_k(N/N))^\top$, and let $\widetilde{\phi}_k(x) = N^{-1}\mathbf{B}(x)^\top \mathbf{V}_{n,p}^{-1}\mathbf{B}^\top \boldsymbol{\phi}_k$ be the B-spline smoothing of $\phi_k(x)$. The linearity of spline smoothing implies that

$$\widetilde{\eta}_i(x) - \eta_i(x) = \widetilde{m}(x) - m(x) + \sum_{k=1}^\infty \xi_{ik}\left\{\widetilde{\phi}_k(x) - \phi_k(x)\right\}.$$

Lemma A.4 in Cao et al (2012) assures there exists a constant $C_{q,\mu} > 0$, such that

$$\|\widetilde{m} - m\|_\infty \le C_{q,\mu}\|m\|_{q,\mu} J_s^{-p^*}, \tag{S.14}$$

$$\|\widetilde{\phi}_k - \phi_k\|_\infty \le C_{q,\mu}\|\phi_k\|_{q,\mu} J_s^{-p^*}, \quad k \ge 1 \tag{S.15}$$

Thus, with norm inequality, we have

$$\|\widetilde{\eta}_i - \eta_i\|_\infty \le \|\widetilde{m} - m\|_\infty + \sum_{k=1}^\infty |\xi_{ik}|\|\widetilde{\phi}_k - \phi_k\|_\infty \le C_{q,\mu} W_i J_s^{-p^*}, \tag{S.16}$$

where $W_i = \|m\|_{q,\mu} + \sum_{k=1}^\infty |\xi_{ik}|\,\|\phi_k\|_{q,\mu}$, $i = 1, \ldots, n$, are i.i.d. nonnegative random variables. According to assumption (A4) and (A5'), $W_i^{r_0}$ has a finite absolute moment and we have

$$\mathbb{P}\left\{\max_{1\le i\le n} W_i > (n\log n)^{2/r_0}\right\} \le n\frac{\mathbb{E}W_i^{r_0}}{(n\log n)^2} = \mathbb{E}W_i^{r_0}(n\log n)^{-2},$$

which implies

$$\sum_{n=1}^\infty \mathbb{P}\left\{\max_{1\le i\le n} W_i > (n\log n)^{2/r_0}\right\} \le \mathbb{E}W_i^{r_0}\sum_{n=1}^\infty (n\log n)^{-2} < +\infty.$$

Thus, according to Borel Cantelli lemma, $\max_{1\le i\le n} W_i = \mathcal{O}_{a.s.}\left\{(n\log n)^{2/r_0}\right\}$ which, together with (S.16) and (S.2), prove the lemma.

The next result bounds the truncated tails of the infinite moving average.

Lemma S.5 *Under Assumption (A5'), as $n \to \infty$*

$$\max_{1\le i\le n}\max_{1\le j\le N}\left|\sum_{t=t_N+1}^{\infty} a_{it}\zeta_{i,j-t}\right| = \mathcal{O}_{a.s.}\left(N^{-5}\right).$$

PROOF Given that $|a_{it}| \le a\rho^t$, one has

$$\mathbb{E}\left|\sum_{t=t_N+1}^{\infty} a_{it}\zeta_{i,j-t}\right| \le \sum_{t=t_N+1}^{\infty}|a_{it}|\,\mathbb{E}\left|\zeta_{i,j-t}\right| \le \mathbb{E}\left|\zeta_{i1}\right|\sum_{t=t_N+1}^{\infty}|a_{it}| \le C\rho^{t_N}.$$

The choice of t_N ensures that the right side of the above is $\le CN^{-10}$. Thus

$$\mathbb{P}\left(\max_{1\le i\le n}\max_{1\le j\le N}\left|\sum_{t=t_N+1}^{\infty} a_{it}\zeta_{i,j-t}\right| \ge N^{-5}\right) \le n\times N\times\frac{CN^{-10}}{N^{-5}} \le CN^{-4+\theta} \le CN^{-4+2\nu} \le CN^{-2}$$

because $\theta < 2\nu$ by (8) and $\nu \le 1$ by definition. Borel-Cantelli lemma then proves the lemma.

Lemma S.6 *Assumptions (A3), (A5') imply Assumption (A5) and*

$$\mathbb{P}\left\{\max_{1\le i\le n}\max_{0\le t\le t_N}\max_{1\le j\le N}\left|\sum_{j'=1}^{j}\zeta_{i,j'-t}-\sum_{j'=1}^{j}U_{i,j'-t,\zeta}\right| > 2N^{*\beta_2}\right\} \le C\left(N^*\right)^{-\gamma_2}.$$

PROOF. Under Assumption (A5'), $\mathbb{E}\left|\xi_{ik}\right|^{r_1} < +\infty$, $r_1 > 4+2\omega$, with ω in Assumption (A4), so there exists $\beta_1 \in (0,1/2)$, such that $r_1 > (2+\omega)/\beta_1$. Denote $H(x) = x^{r_1}$, then Lemma S.1 entails that for each $1 \le k < \infty$, there exist constants c_{1k} and a_k depending on the distribution of ξ_{ik}, such that for $x_n = n^{\beta_1}$, $n/H\left(a_k x_n\right) = a_k^{-r_1}n^{1-r_1\beta_1}$ and i.i.d. $N(0,1)$ variables $\left\{U_{ik,\xi}\right\}_{i=1}^{n}$ on new probability space $\left(\tilde{\Omega}_{k,\xi},\tilde{\mathcal{A}}_{k,\xi},\tilde{\mathbb{P}}_{k,\xi}\right)$

$$\tilde{\mathbb{P}}_{k,\xi}\left\{\max_{1\le t\le n}\left|\sum_{i=1}^{t}\tilde{\xi}_{ik}-\sum_{i=1}^{t}U_{ik,\xi}\right| > n^{\beta_1}\right\} < c_{1k}a_k^{-r_1}n^{1-r_1\beta_1},$$

where $\left\{\tilde{\xi}_{ik}\right\}_{i=1}^{n}$ are defined on $\left(\tilde{\Omega}_{k,\xi},\tilde{\mathcal{A}}_{k,\xi},\tilde{\mathbb{P}}_{k,\xi}\right)$ and equal in distribution to $\left\{\xi_{ik}\right\}_{i=1}^{n}$. Likewise, under Assumption (A5'), $\mathbb{E}\left|\zeta_{i,j}\right|^{r_2} < +\infty$, $r_2 > 4+2\theta$, where θ is defined in Assumption (A3), so there exists some $\beta_2 \in (0,1/2)$, such that $r_2 > (2+\theta)/\beta_2$. For $N^* = N + t_N$

given in Assumption (A5), and $H(x) = x^{r_2}, x_{N^*} = N^{*\beta_2}$, it is easy to check that with $j = 1 - t_N, \ldots, N$ playing the role of $i = 1, \ldots, n$ the condition of Lemma S.1 is satisfied under Assumption (A5). Hence for each $i \in \{1, \ldots, n\}$, according to Lemma S.1, there exist positive constants c_i and b_i depending on the distribution of $\zeta_{i,j}$, and i.i.d. $N(0,1)$ variables $U_{i,j,\zeta}$ on new probability space $\left(\tilde{\Omega}_{i,\zeta}, \tilde{\mathcal{A}}_{i,\zeta}, \tilde{\mathbb{P}}_{i,\zeta}\right)$ such that

$$\tilde{\mathbb{P}}_{i,\zeta}\left\{\max_{1-t_N \le j \le N}\left|\sum_{j'=1-t_N}^{j} \tilde{\zeta}_{i,j'} - \sum_{j'=1-t_N}^{j} U_{i,j',\zeta}\right| > x_{N^*}\right\} < c_i N^*/H\left(b_i x_{N^*}\right) \le C_i N^{*1-r_2\beta_2},$$

where $\left\{\tilde{\zeta}_{i,j'}\right\}_{1-t_N}^{N}$ are defined on $\left(\tilde{\Omega}_{i,\zeta}, \tilde{\mathcal{A}}_{i,\zeta}, \tilde{\mathbb{P}}_{i,\zeta}\right)$ and equal in distribution to $\left\{\zeta_{i,j'}\right\}_{1-t_N}^{N}$.

Since Assumption (A5') stipulates the independence of $\{\xi_{ik}\}_{i=1,k=1}^{n,\infty}$ and $\left\{\zeta_{i,j'}\right\}_{i=1,j'=1-t_N}^{n,N}$, this independence is automatically preserved for $\left\{\tilde{\xi}_{ik}\right\}_{i=1,k=1}^{n,\infty}$ and $\left\{\tilde{\zeta}_{i,j'}\right\}_{i=1,j'=1-t_N}^{n,N}$ if their new probability spaces $\left(\tilde{\Omega}_{k,\xi}, \tilde{\mathcal{A}}_{k,\xi}, \tilde{\mathbb{P}}_{k,\xi}\right), k \ge 1$ and $\left(\tilde{\Omega}_{i,\zeta}, \tilde{\mathcal{A}}_{i,\zeta}, \tilde{\mathbb{P}}_{i,\zeta}\right), i \ge 1$ are all independently embedded into a product probability space

$$\left(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbb{P}}\right) \equiv$$

$$\left(\left(\bigotimes_{k=1}^{\infty} \tilde{\Omega}_{k,\xi}\right) \bigotimes \left(\bigotimes_{i=1}^{\infty} \tilde{\Omega}_{i,\zeta}\right), \left(\bigotimes_{k=1}^{\infty} \tilde{\mathcal{A}}_{k,\xi}\right) \bigotimes \left(\bigotimes_{i=1}^{\infty} \tilde{\mathcal{A}}_{i,\zeta}\right), \tilde{\mathbb{P}}\right)$$

according to Ionescu–Tulcea Theorem (Theorem 14.32 in Klenke 2014). This independent embedding also ensures that all $N(0,1)$ variables $\{U_{ik,\xi}\}_{i=1,k=1}^{n,\infty}$ $\{U_{i,j,\zeta}\}_{i=1,j=1-t_N}^{n,N}$ remain independent in the new product probability space, hence $\{U_{ik,\xi}\}_{i=1,k=1}^{n,\infty}$ and $\{U_{i,j,\zeta}\}_{i=1,j=1-t_N}^{n,N}$ are i.i.d. $N(0,1)$ as required in Assumption (A5).

In what follows, with some abuse of notations, we will not distinguish $\xi_{ik}, \zeta_{i,j}$ on the original probability space from $\tilde{\xi}_{ik}, \tilde{\zeta}_{i,j}$ on the above product probability space, nor the original probability measure $\mathbb{P}$ from $\tilde{\mathbb{P}}$ on the product space.

As the number of distinct distributions for ξ_{ik} is finite, there is a common $c > 0$, such that $\mathbb{P}\left\{\max_{1\le t\le n}\left|\sum_{i=1}^{t} \xi_{ik} - \sum_{i=1}^{t} U_{ik,\xi}\right| > n^{\beta_1}\right\} < cn^{1-r_1\beta_1}$, for each $1 \le k < \infty$. Consequently, there is a $c_1 > 0$ such that

$$\mathbb{P}\left\{\max_{1\le k\le k_n}\max_{1\le t\le n}\left|\sum_{i=1}^{t} \xi_{ik} - \sum_{i=1}^{t} U_{ik,\xi}\right| > n^{\beta_1}\right\} < k_n c n^{1-r_1\beta_1} \le c_1 n^{-\gamma_1},$$

where $\gamma_1 = r_1\beta_1 - 1 - \omega > 1$ by noticing that $r_1 > (2+\omega)/\beta_1$.

Since the number of distinct distributions for $\left\{\zeta_{i,1}\right\}_{i\geq 1}$ is finite, there exists a constant c such that for all $i \in \{1,\ldots,n\}$

$$\mathbb{P}\left\{\max_{1-t_N\leq j\leq N}\left|\sum_{j'=1-t_N}^{j}\zeta_{i,j'} - \sum_{j'=1-t_N}^{j}U_{i,j',\zeta}\right| > x_{N^*}\right\} < cN^{*1-r_2\beta_2}.$$

Thus

$$\mathbb{P}\left\{\max_{1\leq i\leq n}\max_{1-t_N\leq j\leq N}\left|\sum_{j'=1-t_N}^{j}\zeta_{i,j'} - \sum_{j'=1-t_N}^{j}U_{i,j',\zeta}\right| > x_{N^*}\right\} < cnN^{*1-r_2\beta_2} \leq c_2\left(N^*\right)^{1-r_2\beta_2+\theta}$$

Let $\gamma_2 = r_2\beta_2 - \theta - 1$, Since $r_2 > (2+\theta)/\beta_2$ according to the above discussion, it is easy to see that $\gamma_2 > 1$, so the Assumption (A5) is proved.

Next for $t = 0,\ldots,t_N$

$$\begin{aligned}
&\max_{1\leq j\leq N}\left|\sum_{j'=1}^{j}\zeta_{i,j'} - \sum_{j'=1}^{j}U_{i,j'-t,\zeta}\right| = \\
&\max_{1\leq j\leq N}\left|\left(\sum_{j'=1-t_N}^{j-t}\zeta_{i,j'} - \sum_{j'=1-t_N}^{j-t}U_{i,j',\zeta}\right) - \left(\sum_{j'=1-t_N}^{-t}\zeta_{i,j'} - \sum_{j'=1-t_N}^{-t}U_{i,j',\zeta}\right)\right| \\
&\leq \max_{1\leq j\leq N}\left|\sum_{j'=1-t_N}^{j-t}\zeta_{i,j'} - \sum_{j'=1-t_N}^{j-t}U_{i,j',\zeta}\right| + \left|\sum_{j'=1-t_N}^{-t}\zeta_{i,j'} - \sum_{j'=1-t_N}^{-t}U_{i,j',\zeta}\right| \\
&\leq 2\max_{1-t_N\leq j\leq N}\left|\sum_{j'=1-t_N}^{j}\zeta_{i,j'} - \sum_{j'=1-t_N}^{j}U_{i,j',\zeta}\right|
\end{aligned}$$

Thus

$$\mathbb{P}\left\{\max_{0\le t\le t_N}\max_{1\le j\le N}\left|\sum_{j'=1}^{j}\zeta_{i,j'-t}-\sum_{j'=1}^{j}U_{i,j'-t,\zeta}\right|>2N^{*\beta_2}\right\}$$
$$\le\mathbb{P}\left\{2\max_{1-t_N\le j\le N}\left|\sum_{j'=1-t_N}^{j}\zeta_{i,j'}-\sum_{j'=1-t_N}^{j}U_{i,j',\zeta}\right|>2N^{*\beta_2}\right\}$$
$$=\mathbb{P}\left\{\max_{1-t_N\le j\le N}\left|\sum_{j'=1-t_N}^{j}\zeta_{i,j'}-\sum_{j'=1-t_N}^{j}U_{i,j',\zeta}\right|>N^{*\beta_2}\right\}\le CN^{*(1-r_2\beta_2)}.$$

Bonferroni technique then leads to

$$\mathbb{P}\left\{\max_{1\le i\le n}\max_{0\le t\le t_N}\max_{1\le j\le N}\left|\sum_{j'=1}^{j}\zeta_{i,j'-t}-U_{i,j'-t,\zeta}\right|>2N^{*\beta_2}\right\}\le CnN^{*(1-r_2\beta_2)}\le C\left(N^*\right)^{-\gamma_2}.$$

The lemma holds consequently.

Lemma S.7 *Under Assumptions (A2), (A6), as* $n\to\infty$

$$\max_{1\le i\le n}\max_{1\le \ell\le J_s+p}\max_{0\le t\le t_N}\left|N^{-1}\sum\nolimits_{j=1}^{N}B_{\ell,p}\left(\frac{j}{N}\right)\sigma_i\left(\frac{j}{N}\right)\zeta_{i,j-t}\right|$$
$$=\mathcal{O}_{a.s}\left(N^{-1/2}J_s^{-1/2}\sqrt{\log N}\right)+\mathcal{O}_{a.s}\left(N^{\beta_2-1}\right),\tag{S.17}$$

$$\max_{1\le \ell\le J_s+p}\max_{0\le t\le t_N}\left|n^{-1}\sum\nolimits_{i=1}^{n}N^{-1}\sum\nolimits_{j=1}^{N}B_{\ell,p}\left(\frac{j}{N}\right)\sigma\left(\frac{j}{N}\right)\zeta_{i,j-t}\right|$$
$$=\mathcal{O}_{a.s}\left(N^{-1/2}n^{-1/2}J_s^{-1/2}\sqrt{\log N}\right)+\mathcal{O}_{a.s}\left(N^{\beta_2-1}\right).\tag{S.18}$$

Proof

$$\max_{1\le i\le n}\max_{1\le \ell\le J_s+p}\max_{0\le t\le t_N}\left|N^{-1}\sum\nolimits_{j=1}^{N}B_{\ell,p}\left(\frac{j}{N}\right)\sigma_i\left(\frac{j}{N}\right)\zeta_{i,j-t}\right|$$
$$\le\max_{1\le i\le n}\max_{1\le \ell\le J_s+p}\max_{0\le t\le t_N}\left|N^{-1}\sum\nolimits_{j=1}^{N}B_{\ell,p}\left(\frac{j}{N}\right)\sigma_i\left(\frac{j}{N}\right)U_{i,j-t,\zeta}\right|+$$
$$\max_{1\le i\le n}\max_{1\le \ell\le J_s+p}\max_{0\le t\le t_N}\left|N^{-1}\sum\nolimits_{j=1}^{N}B_{\ell,p}\left(\frac{j}{N}\right)\sigma_i\left(\frac{j}{N}\right)\left(\zeta_{i,j-t}-U_{i,j-t,\zeta}\right)\right|.$$

Where $U_{i,j,\zeta}$ is defined in Assumption (A5). For the first part of equality above,

$$\mathbb{E}\left\{N^{-1}\sum\nolimits_{j=1}^{N} B_{\ell,p}\left(\frac{j}{N}\right)\sigma_i\left(\frac{j}{N}\right)U_{i,j-t,\zeta}\right\}^2 =$$
$$N^{-2}\sum\nolimits_{j=1}^{N} B_{\ell,p}^2\left(\frac{j}{N}\right)\sigma_i^2\left(\frac{j}{N}\right)\mathbb{E}U_{i,j-t,\zeta}^2 \leq \frac{C}{NJ_s}$$

$$\mathbb{P}\left\{\frac{\max_{1\leq i\leq n}\max_{1\leq \ell\leq J_s+p}\max_{0\leq t\leq t_N}\left|N^{-1}\sum_{j=1}^{N} B_{\ell,p}\left(\frac{j}{N}\right)\sigma_i\left(\frac{j}{N}\right)U_{i,j-t,\zeta}\right|}{\left[\mathbb{E}\left\{N^{-1}\sum_{j=1}^{N} B_{\ell,p}\left(\frac{j}{N}\right)\sigma_i\left(\frac{j}{N}\right)U_{i,j-t,\zeta}\right\}^2\right]^{1/2}} \geq 4\sqrt{\log N}\right\}$$
$$\leq t_N n(J_s+p)\mathbb{P}\left\{\frac{\left|N^{-1}\sum_{j=1}^{N} B_{\ell,p}\left(\frac{j}{N}\right)\sigma_i\left(\frac{j}{N}\right)U_{i,j-t,\zeta}\right|}{\left[\mathbb{E}\left\{N^{-1}\sum_{j=1}^{N} B_{\ell,p}\left(\frac{j}{N}\right)\sigma_i\left(\frac{j}{N}\right)U_{i,j-t,\zeta}\right\}^2\right]^{1/2}} \geq 4\sqrt{\log N}\right\}$$
$$\leq t_N n(J_s+p)\exp\left\{\frac{-4^2\log N}{2}\right\} \leq Ct_N n(J_s+p)N^{-8} \leq CN^{-8+\theta+\gamma}\log N \leq CN^{-5},$$

because $\theta < 2\nu$ by (8) and $\nu \leq 1$ by definition, and $\gamma < 1-\beta_2-\theta/2 < 1$ by (11). Thus Borel-Cantelli lemma entails that

$$\max_{1\leq i\leq n}\max_{1\leq \ell\leq J_s+p}\max_{0\leq t\leq t_N}\left|N^{-1}\sum\nolimits_{j=1}^{N} B_{\ell,p}\left(\frac{j}{N}\right)\sigma_i\left(\frac{j}{N}\right)U_{i,j-t,\zeta}\right|$$
$$= \mathcal{O}_{a.s}\left(N^{-1/2}J_s^{-1/2}\sqrt{\log N}\right).$$

The counterpart for averaging over $i = 1, \ldots, n$ is worked out as

$$\mathbb{P}\left\{\frac{\max_{1\leq \ell\leq J_s+p}\max_{0\leq t\leq t_N}\left|n^{-1}\sum_{i=1}^{n} N^{-1}\sum_{j=1}^{N} B_{\ell,p}\left(\frac{j}{N}\right)\sigma_i\left(\frac{j}{N}\right)U_{i,j-t,\zeta}\right|}{\left\{\mathbb{E}\left\{n^{-1}\sum_{i=1}^{n} N^{-1}\sum_{j=1}^{N} B_{\ell,p}\left(\frac{j}{N}\right)\sigma_i\left(\frac{j}{N}\right)U_{i,j-t,\zeta}\right\}^2\right\}^{1/2}} \geq 4\sqrt{\log N}\right\}$$
$$\leq (J_s+p)t_N\mathbb{P}\left\{\frac{\left|n^{-1}\sum_{i=1}^{n} N^{-1}\sum_{j=1}^{N} B_{\ell,p}\left(\frac{j}{N}\right)\sigma_i\left(\frac{j}{N}\right)U_{i,j-t,\zeta}\right|}{\left\{\mathbb{E}\left\{n^{-1}\sum_{i=1}^{n} N^{-1}\sum_{j=1}^{N} B_{\ell,p}\left(\frac{j}{N}\right)\sigma_i\left(\frac{j}{N}\right)U_{i,j-t,\zeta}\right\}^2\right\}^{1/2}} \geq 4\sqrt{\log N}\right\}$$
$$\leq (J_s+p)t_N\exp\left\{\frac{-4^2\log N}{2}\right\} \leq C(J_s+p)N^{-8} \leq CN^{-7},$$

while

$$\mathbb{E}\left\{n^{-1}\sum\nolimits_{i=1}^{n}N^{-1}\sum\nolimits_{j=1}^{N}B_{\ell,p}\left(\frac{j}{N}\right)\sigma_i\left(\frac{j}{N}\right)U_{i,j-t,\zeta}\right\}^2$$

$$(nN)^{-2}\sum\nolimits_{i=1}^{n}\sum\nolimits_{j=1}^{N}B_{\ell,p}^2\left(\frac{j}{N}\right)\sigma_i^2\left(\frac{j}{N}\right)\mathbb{E}U_{i,j-t,\zeta}^2\leq\frac{C}{nNJ_s}$$

Notice next that

$$\sum\nolimits_{j=1}^{N}B_{\ell,p}\left(\frac{j}{N}\right)\sigma_i\left(\frac{j}{N}\right)\left(\zeta_{i,j-t}-U_{i,j-t,\zeta}\right)$$

$$=\sum_{j=1}^{N-1}\left\{B_{\ell,p}\left(\frac{j}{N}\right)\sigma_i\left(\frac{j}{N}\right)-B_{\ell,p}\left(\frac{j+1}{N}\right)\sigma_i\left(\frac{j+1}{N}\right)\right\}\sum_{j'=1}^{j}\left(\zeta_{i,j'-t}-U_{i,j'-t,\zeta}\right)$$

$$+B_{\ell,p}(1)\,\sigma_i(1)\sum_{j'=1}^{N}\left(\zeta_{i,j'-t}-U_{i,j'-t,\zeta}\right)$$

While

$$\left|\sum_{j=1}^{N-1}\left\{B_{\ell,p}\left(\frac{j}{N}\right)\sigma_i\left(\frac{j}{N}\right)-B_{\ell,p}\left(\frac{j+1}{N}\right)\sigma_i\left(\frac{j+1}{N}\right)\right\}\sum_{j'=1}^{j}\left(\zeta_{i,j'-t}-U_{i,j'-t,\zeta}\right)\right|$$

$$\leq\sum_{j=1}^{N-1}\left|B_{\ell,p}\left(\frac{j}{N}\right)\sigma_i\left(\frac{j}{N}\right)-B_{\ell,p}\left(\frac{j+1}{N}\right)\sigma_i\left(\frac{j+1}{N}\right)\right|\left|\sum_{j'=1}^{j}\left(\zeta_{i,j'-t}-U_{i,j'-t,\zeta}\right)\right|$$

$$\leq CNJ_s^{-1}\left(N^{-\nu}+J_sN^{-1}\right)\max_{1\leq j\leq N}\left|\sum_{j'=1}^{j}\left(\zeta_{i,j'-t}-U_{i,j'-t,\zeta}\right)\right|$$

Consequently, according to (S.3),

$$\max_{1\leq i\leq n}\max_{1\leq\ell\leq J_s+p}\max_{0\leq t\leq t_N}\left|N^{-1}\sum\nolimits_{j=1}^{N}B_{\ell,p}\left(\frac{j}{N}\right)\sigma_i\left(\frac{j}{N}\right)\left(\zeta_{i,j-t}-U_{i,j-t,\zeta}\right)\right|$$

$$\leq N^{-1}C\left(NJ_s^{-1}N^{-\nu}+1\right)\max_{0\leq t\leq t_N}\max_{1\leq i\leq n}\max_{1\leq j\leq N}\left|\sum_{j'=1}^{j}\left(\zeta_{i,j'-t}-U_{i,j'-t,\zeta}\right)\right|$$

According to Lemma S.3, the inequality above becomes

$$\leq CN^{-1}\max_{0\leq t\leq t_N}\max_{1\leq i\leq n}\max_{1\leq j\leq N}\left|\sum_{j'=1}^{j}\left(\zeta_{i,j'-t}-U_{i,j'-t,\zeta}\right)\right|=\mathcal{O}_{a.s}\left(N^{-1}N^{*\beta_2}\right)=\mathcal{O}_{a.s}\left(N^{\beta_2-1}\right)$$

Combine the results above, the lemma is proved.

Lemma S.8 *Under Assumptions (A5) and (A6), as $n \to \infty$*

$$\max_{1\le i\le n} \|\widetilde{\varepsilon}_i\|_\infty = \max_{1\le i\le n} \sup_{x\in[0,1]} |\widetilde{\varepsilon}_i(x)| = \mathcal{O}_{a.s}\left(N^{-1/2} J_s^{1/2}\sqrt{\log N} + N^{\beta_2-1}J_s\right). \tag{S.19}$$

$$\left\|n^{-1}\sum\nolimits_{i=1}^n \widetilde{\varepsilon}_i\right\|_\infty = \mathcal{O}_{a.s}\left(N^{-1/2}n^{-1/2}J_s^{1/2}\sqrt{\log N}\right) + \mathcal{O}_{a.s}\left(N^{\beta_2-1}J_s\right) = o_{a.s.}\left(n^{-1/2}\right). \tag{S.20}$$

PROOF One first writes $\widetilde{\varepsilon}_i(x)$ as a finite sum and an infinite remainder

$$\widetilde{\varepsilon}_i(x) = \sum_{t=0}^\infty a_{it}\widetilde{\zeta}_{i,-t}(x) = Q_{1i}\left(x\right) + Q_{2i}\left(x\right),$$

in which

$$Q_{1i}\left(x\right) = \sum_{t=0}^{t_N} a_{it}\widetilde{\zeta}_{i,-t}(x), Q_{2i}\left(x\right) = \sum_{t=t_N+1}^{\infty} a_{it}\widetilde{\zeta}_{i,-t}(x), \tag{S.21}$$

with

$$\widetilde{\zeta}_{i,-t}\left(x\right) = N^{-1}\mathbf{B}(x)^\top \mathbf{V}_{n,p}^{-1}\mathbf{B}^\top\left(\sigma_i\left(j/N\right)\zeta_{i,j-t}\right)_{1\le j\le N}^\top.$$

The average $n^{-1}\sum_{i=1}^n \widetilde{\varepsilon}_i(x)$ is likewise decomposed as

$$n^{-1}\sum\nolimits_{i=1}^n \widetilde{\varepsilon}_i(x) = Q_1\left(x\right) + Q_2\left(x\right),$$

where

$$Q_1\left(x\right) = n^{-1}\sum\nolimits_{i=1}^n Q_{1i}\left(x\right), Q_2\left(x\right) = n^{-1}\sum\nolimits_{i=1}^n Q_{2i}\left(x\right),$$

with $Q_{1i}\left(x\right), Q_{2i}\left(x\right)$ defined in (S.21). The terms Q_{2i}, Q_2, Q_{1i}, Q_1 are then bounded separately.

To begin with, note that

$$\begin{aligned}\left|Q_{2i}\left(x\right)\right| &= \left|N^{-1}\mathbf{B}(x)^\top\mathbf{V}_{n,p}^{-1}\mathbf{B}^\top\left(\sum_{t=t_N+1}^\infty a_{it}\left(\sigma_i\left(j/N\right)\zeta_{i,j-t}\right)_{1\le j\le N}^\top\right)\right| \\ &\le \left|\mathbf{B}(x)^\top\mathbf{V}_{n,p}^{-1}N^{-1}\mathbf{B}^\top\mathbf{1}_N\right| \|\sigma_i\|_\infty \max_{1\le i\le n}\max_{1\le j\le N}\left|\sum_{t=t_N+1}^\infty a_{it}\zeta_{i,j-t}\right|.\end{aligned}$$

According to Lemma A.3 in Cao et al (2012), $\left|\mathbf{V}_{n,p}^{-1}\right|_{\infty} \leq M_p J_s$. Since the B-spline basis $\|\mathbf{B}(x)\|_{\infty} \leq 1$ so $\left\|N^{-1}\mathbf{B}^{\top}\mathbf{1}_N\right\|_{\infty} \leq 1$, and Lemma S.5 ensures that

$$\max_{1\leq i\leq n}\max_{1\leq j\leq N}\left|\sum_{t=t_N+1}^{\infty} a_{it}\zeta_{i,j-t}\right| = \mathcal{O}_{a.s.}\left(N^{-5}\right)$$

one obtains that

$$\begin{aligned}\max_{1\leq i\leq n}\|Q_{2i}\|_{\infty} = \max_{1\leq i\leq n}\sup_{x\in[0,1]}|Q_{2i}(x)| &\leq \sup_{x\in[0,1]}\left|\left|\mathbf{B}(x)^{\top}\mathbf{V}_{n,p}^{-1}\right|_{\infty}(J_s+p)\right| \times \mathcal{O}_{a.s.}\left(N^{-5}\right)\\ &\leq |M_p J_s(J_s+p)| \times \mathcal{O}_{a.s.}\left(N^{-5}\right) = \mathcal{O}_{a.s.}\left(N^{-3}\right). \qquad \text{(S.22)}\end{aligned}$$

One obtains immediately from (S.22) that

$$\|Q_2\|_{\infty} = \left\|n^{-1}\sum\nolimits_{i=1}^{n} Q_{2i}\right\|_{\infty} \leq \max_{1\leq i\leq n}\|Q_{2i}\|_{\infty} = \mathcal{O}_{a.s.}\left(N^{-3}\right) = o_{a.s}\left(n^{-1/2}\right).$$

Meanwhile

$$\begin{aligned}\max_{1\leq i\leq n}\sup_{x\in[0,1]}|Q_{1i}(x)| &= \max_{1\leq i\leq n}\sup_{x\in[0,1]}\left|\sum_{t=0}^{t_N} a_{it}N^{-1}\mathbf{B}(x)^{\top}\mathbf{V}_{n,p}^{-1}\mathbf{B}^{\top}\left(\sigma_i(j/N)\zeta_{i,j-t}\right)_{1\leq j\leq N}^{\top}\right|\\ &\leq \max_{1\leq i\leq n}\sup_{x\in[0,1]}\left|\sum_{t=0}^{t_N} a\rho^{t-1}\mathbf{B}(x)^{\top}\left(\mathbf{V}_{n,p}^{-1}N^{-1}\mathbf{B}^{\top}\left(\sigma_i(j/N)\zeta_{i,j-t}\right)_{1\leq j\leq N}^{\top}\right)\right|\end{aligned}$$

$$\begin{aligned}&\leq \max_{1\leq i\leq n}\sum_{t=0}^{t_N} a\rho^{t-1}p\left\|\mathbf{V}_{n,p}^{-1}N^{-1}\mathbf{B}^{\top}\left(\sigma_i(j/N)\zeta_{i,j-t}\right)_{1\leq j\leq N}^{\top}\right\|_{\infty}\\ &\leq \max_{1\leq i\leq n}\sum_{t=0}^{t_N} a\rho^{t-1}p\left|\mathbf{V}_{n,p}^{-1}\right|_{\infty}\max_{1\leq \ell\leq J_s+p}\max_{0\leq t\leq t_N}\left|N^{-1}\sum\nolimits_{j=1}^{N} B_{\ell,p}\left(\frac{j}{N}\right)\sigma_i\left(\frac{j}{N}\right)\zeta_{i,j-t}\right|\end{aligned}$$

$$\begin{aligned}&\leq CJ_s\max_{1\leq i\leq n}\max_{1\leq \ell\leq J_s+p}\max_{0\leq t\leq t_N}\left|N^{-1}\sum\nolimits_{j=1}^{N} B_{\ell,p}\left(\frac{j}{N}\right)\sigma_i\left(\frac{j}{N}\right)\zeta_{i,j-t}\right|\\ &= \mathcal{O}_{a.s}\left(N^{-1/2}J_s^{1/2}\sqrt{\log N}\right) + \mathcal{O}_{a.s}\left(N^{\beta_2-1}J_s\right)\end{aligned}$$

according to (S.17).

Likewise

$$\sup_{x\in[0,1]}|Q_1(x)| = \sup_{x\in[0,1]}\left|n^{-1}\sum\nolimits_{i=1}^{n}\sum_{t=0}^{t_N} a_{it}N^{-1}\mathbf{B}(x)^{\top}\mathbf{V}_{n,p}^{-1}\mathbf{B}^{\top}\left(\sigma_i\left(j/N\right)\zeta_{i,j-t}\right)_{1\le j\le N}^{\top}\right|$$

$$\le CJ_s \max_{1\le \ell\le J_s+p}\max_{0\le t\le t_N}\left|n^{-1}\sum\nolimits_{i=1}^{n} N^{-1}\sum\nolimits_{j=1}^{N} B_{\ell,p}\left(\frac{j}{N}\right)\sigma_i\left(\frac{j}{N}\right)\zeta_{i,j-t}\right|.$$

Applying (S.18) in Lemma S.7, one obtains that

$$\begin{aligned}&\max_{1\le \ell\le J_s+p}\max_{0\le t\le t_N}\left|n^{-1}\sum\nolimits_{i=1}^{n} N^{-1}\sum\nolimits_{j=1}^{N} B_{\ell,p}\left(\frac{j}{N}\right)\sigma_i\left(\frac{j}{N}\right)\zeta_{i,j-t}\right|\\ =\ &\mathcal{O}_{a.s}\left(n^{-1/2}N^{-1/2}J_s^{-1/2}\sqrt{\log N}\right)+\mathcal{O}_{a.s}\left(N^{\beta_2-1}\right)\end{aligned}$$

and together with (S.4) and (S.5),

$$\begin{aligned}&\sup_{x\in[0,1]}|Q_1(x)|\\ &=\mathcal{O}_{a.s}\left(N^{-1/2}n^{-1/2}J_s^{1/2}\sqrt{\log N}\right)+\mathcal{O}_{a.s}\left(N^{\beta_2-1}J_s\right)=o_{a.s}\left(n^{-1/2}\right).\end{aligned}$$

The lemma is proved.

S.2 Proof of Theorem 1

Following the decomposition Equation (S.13), it suffices to bound term $n^{-1}\sum_{i=1}^{n}\{\widehat{\eta}_i(\cdot)-\eta_i(\cdot)\}$ and term $n^{-1}\sum_{i=1}^{n}\widetilde{\varepsilon}_i(\cdot)$ separately. Lemma S.4 has shown that $\max_{1\le i\le n}\|\widetilde{\eta}_i-\eta_i\|_{\infty}=\mathcal{O}_{a.s.}\left\{J_s^{-p^*}(n\log n)^{2/r_0}\right\}=o_{a.s.}\left(n^{-1/2}\right)$. On the other hand, Equation (S.20) shows that $\sup_{x\in(0,1)}|n^{-1}\sum_{i=1}^{n}\widetilde{\varepsilon}_i(x)|=o_{a.s.}\left(n^{-1/2}\right)$. Putting results together, one obtains

$$\begin{aligned}\sup_{x\in[0,1]} n^{1/2}\left|\widehat{m}(x)-\overline{m}(x)\right| &\le n^{1/2}\max_{1\le i\le n}\|\widetilde{\eta}_i-\eta_i\|_{\infty}+n^{1/2}\sup_{x\in(0,1)}\left|n^{-1}\sum\nolimits_{i=1}^{n}\widetilde{\varepsilon}_i(x)\right|\\ &= o_{a.s.}(1).\end{aligned}$$

S.3 Proof of Theorem 2

Denote a random process $\varsigma(x)$ as

$$\varsigma(x) = n^{1/2} G(x,x)^{-1/2} \sum_{k=1}^{\infty} \overline{U}_{.k,\xi} \phi_k(x),$$

where $\overline{U}_{.k,\xi} = n^{-1} \sum_{i=1}^{n} U_{ik,\xi}$, with i.i.d. $N(0,1)$ variables $U_{ik,\xi}, 1 \leq i \leq n, 1 \leq k < \infty$ given in Assumption (A5). Consequently, $n^{1/2}\overline{U}_{.k,\xi}$ are i.i.d. $N(0,1)$, $1 \leq k < \infty$, so $\varsigma(x)$ is a standardized Gaussian process (i.e., mean zero, variance one) with covariance function $\mathbb{E}\varsigma(x)\varsigma(x') = G(x,x)^{-1/2} G(x',x')^{-1/2} G(x,x')$, for any $x, x' \in [0,1]$, thus having the same distribution as $\zeta(x)$.

Notice that $\overline{m}(x) - m(x) = \sum_{k=1}^{\infty} \overline{\xi}_{.k} \phi_k(x)$, in which $\overline{\xi}_{.k} = n^{-1} \sum_{i=1}^{n} \xi_{ik}$. Thus, it suffices to show that

$$\begin{aligned}
&\sup_{x\in[0,1]} \left| n^{1/2} G(x,x)^{-1/2} (\overline{m}(x) - m(x)) - \varsigma(x) \right| \\
&= n^{1/2} \sup_{x\in[0,1]} G(x,x)^{-1/2} \left| \sum_{k=1}^{\infty} \left(\overline{\xi}_{.k} - \overline{U}_{.k,\xi}\right) \phi_k(x) \right| = o_p(1),
\end{aligned}$$

the theorem is then proved with the help of Slutsky's Theorem.

Since $G(x,x)$ has a positive lower bound $C_G > 0$ according to Assumption (A4), one obtains

$$n^{1/2} \sup_{x\in[0,1]} G(x,x)^{-1/2} \left| \sum_{k=1}^{\infty} \left(\overline{\xi}_{.k} - \overline{U}_{.k,\xi}\right) \phi_k(x) \right| \leq C n^{1/2} \sup_{x\in[0,1]} \left| \sum_{k=1}^{\infty} \left(\overline{\xi}_{.k} - \overline{U}_{.k,\xi}\right) \phi_k(x) \right|.$$

Next, using the truncation technique, the right hand side of the inequality can be decomposed into

$$n^{1/2} \sup_{x\in[0,1]} \left| \sum_{k=1}^{\infty} \left(\overline{\xi}_{.k} - \overline{U}_{.k,\xi}\right) \phi_k(x) \right| \leq I_1 + I_2,$$

where

$$I_1 = n^{1/2} \sup_{x\in[0,1]} \sum_{k=1}^{k_n} \left|\overline{\xi}_{.k} - \overline{U}_{.k,\xi}\right| \left|\phi_k(x)\right|,$$

and

$$I_2 = n^{1/2} \sup_{x\in[0,1]} \sum_{k=k_n+1}^{\infty} \left|\overline{\xi}_{.k} - \overline{U}_{.k,\xi}\right| \left|\phi_k(x)\right|.$$

By Assumption (A5)'s strong approximation equation of $\{\xi_{ik}\}$ and Borel-Cantelli lemma, $\max_{1\le k\le k_n} \left|\overline{\xi}_{.k} - \overline{U}_{.k,\xi}\right| = \mathcal{O}_{a.s}\left(n^{\beta_1-1}\right)$. Thus,

$$I_1 \le n^{1/2} \max_{1\le k\le k_n} \left|\overline{\xi}_{.k} - \overline{U}_{.k,\xi}\right| \sup_{x\in[0,1]} \sum_{k=1}^{k_n} \left|\phi_k(x)\right|_\infty = \mathcal{O}_{a.s}\left(n^{\beta_1-1/2}\right) = o_{a.s}(1).$$

On the other hand

$$\begin{aligned}
\mathbb{E}I_2 &\le n^{1/2} \sup_{x\in[0,1]} \sum_{k=k_n+1}^{\infty} \left(\mathbb{E}\left|\overline{\xi}_{.k}\right| + \mathbb{E}\left|\overline{U}_{.k,\xi}\right|\right) \left|\phi_k(x)\right| \\
&\le n^{1/2} \sup_{x\in[0,1]} \sum_{k=k_n+1}^{\infty} \left(\left(\mathbb{E}\left|\overline{\xi}_{.k}\right|^2\right)^{1/2} + \left(\mathbb{E}\left|\overline{U}_{.k,\xi}\right|^2\right)^{1/2}\right) \left|\phi_k(x)\right| \\
&= n^{1/2} \sup_{x\in[0,1]} \sum_{k=k_n+1}^{\infty} \left(\left(n^{-1}\right)^{1/2} + \left(n^{-1}\right)^{1/2}\right) \left|\phi_k(x)\right| \\
&\le \sum_{k=k_n+1}^{\infty} \sup_{x\in[0,1]} \left|\phi_k(x)\right| = \sum_{k=k_n+1}^{\infty} \left\|\phi_k\right\|_\infty = o(1),
\end{aligned}$$

which leads to $I_2 = o_p(1)$.

Combining results above, one obtains

$$\sup_{x\in[0,1]} \left|n^{1/2} G(x,x)^{-1/2} \left(\overline{m}(x) - m(x)\right) - \varsigma(x)\right| = o_p(1),$$

thus for any fixed $x \in [0,1]$

$$\left|n^{1/2} G(x,x)^{-1/2} \left(\overline{m}(x) - m(x)\right) - \varsigma(x)\right| = o_p(1).$$

Applying Slutsky's Theorem,

$$\sup_{x\in[0,1]} \left|n^{1/2} G(x,x)^{-1/2} \left(\overline{m}(x) - m(x)\right)\right| \to \sup_{x\in[0,1]} \left|\varsigma(x)\right|,$$

$$n^{1/2} G(x,x)^{-1/2} \left(\overline{m}(x) - m(x)\right) \to \varsigma(x), x \in [0,1],$$

in distribution.

Consequently, notice that

$$\begin{aligned} & n^{1/2} G(x,x)^{-1/2} (\widehat{m}(x) - m(x)) \\ = \; & n^{1/2} G(x,x)^{-1/2} (\widehat{m}(x) - \overline{m}(x)) + n^{1/2} G(x,x)^{-1/2} (\overline{m}(x) - m(x)), \end{aligned}$$

in which the first term of right hand side is uniformly $o_{a.s}(1)$ according to Theorem 1, thus Banach space valued Slutsky's Theorem (Theorem 2.3, Bosq (2000)) entails that $n^{1/2} G(x,x)^{-1/2} (\widehat{m}(x) - m(x))$ converges weakly to the same stochastic process as $n^{1/2} G(x,x)^{-1/2} (\overline{m}(x) - m(x))$. Hence Continuous Mapping Theorem for Banach space valued random variables (equation (2.11), Bosq (2000)) implies that

$$\sup_{x \in [0,1]} \left| n^{1/2} G(x,x)^{-1/2} (\widehat{m}(x) - m(x)) \right| \to \sup_{x \in [0,1]} |\varsigma(x)|,$$

$$n^{1/2} G(x,x)^{-1/2} (\widehat{m}(x) - m(x)) \to \varsigma(x), x \in [0,1].$$

The Theorem is thus proved.

S.4 Proof of Proposition 1

Note that for $(x, x') \in [0,1]^2$

$$\overline{G}(x,x') = n^{-1} \sum_{i=1}^{n} C_i(x,x'), C_i(x,x') = \{\eta_i(x) - m(x)\} \{\eta_i(x') - m(x')\}, \tag{S.23}$$

where $\{C_i(\cdot,\cdot)\}_{i=1}^{n}$ is an i.i.d. sequence of $\mathcal{C}[0,1]^2$ random variables. By the strong law of large numbers for Banach space valued random sequence (Theorem 2.4, Bosq (2000)), their sample mean $\overline{G}(\cdot,\cdot)$ converges almost surely in Banach space norm to its mean $\mathbb{E}C_1(\cdot,\cdot) \equiv G(\cdot,\cdot)$, thus (14) is proved. Similarly, one obtains that $\|\overline{m} - m\|_\infty = o_{a.s}(1)$ and consequently $\|\widehat{m} - m\|_\infty = o_{a.s}(1)$.

Next, notice that

$$\left(\widehat{G} - \overline{G}\right)(x,x') = n^{-1} \sum_{i=1}^{n} \{I_{1i}(x) I_{2i}(x') + I_{1i}(x') I_{2i}(x) - I_{1i}(x) I_{1i}(x')\},$$

where

$$\begin{aligned} I_{1i}(\cdot) &= \widehat{\eta}_i(\cdot) - \eta_i(\cdot) - \{\widehat{m}(\cdot) - m(\cdot)\} \\ I_{2i}(\cdot) &= \widehat{\eta}_i(\cdot) - \widehat{m}(\cdot) \\ &= \widehat{\eta}_i(\cdot) - \eta_i(\cdot) - \{\widehat{m}(\cdot) - m(\cdot)\} + \eta_i(\cdot) - m(\cdot). \end{aligned}$$

Equation (S.9), Lemmas S.4 and S.8 and Equation (S.4) imply

$$\begin{aligned} \max_i \|\widehat{\eta}_i - \eta_i\|_\infty &\leq \max_i \|\widetilde{\eta}_i(\cdot) - \eta_i(\cdot)\|_\infty + \max_i \|\widetilde{\varepsilon}_i\|_\infty \\ &= o_{a.s}\left(n^{-1/2}\right) + \mathcal{O}_{a.s}\left(N^{-1/2} J_s^{1/2} \sqrt{\log N} + N^{\beta_2 - 1} J_s\right) \\ &= o_{a.s}\left(n^{-1/2}\right) + o_{a.s}\left(N^{-\beta_2/2 - \theta/4}\right) = o_{a.s}(1), \end{aligned} \tag{S.24}$$

which, together with $\|\widehat{m} - m\|_\infty = o_{a.s}(1)$. entail that $\left\|\widehat{G} - \overline{G}\right\|_\infty = o_{a.s}(1)$, that is, (15). Equations (14)-(15) and triangle inequality of $\|\cdot\|_\infty$ lead to (16). The Proposition is thus proved.

S.5 Proof of Corollary 1

Notice that

$$\begin{aligned} & n^{1/2} \widehat{G}(x,x)^{-1/2} (\widehat{m}(x) - m(x)) \\ = \; & n^{1/2} G(x,x)^{-1/2} (\widehat{m}(x) - m(x)) + n^{1/2} \left(\widehat{G}(x,x)^{-1/2} - G(x,x)^{-1/2}\right) (\widehat{m}(x) - m(x)). \end{aligned}$$

Since $G(x,x)$ is uniformly bounded away from 0 by Assumption (A4), one obtains $sup_{x\in[0,1]} \left|\widehat{G}(x,x)^{-1/2} - G(x,x)^{-1/2}\right| = o_{a.s}(1)$ by (16). Meanwhile, Theorem 2 has shown that $n^{1/2}(\widehat{m}(x) - m(x)) \to G^{1/2}(x,x)\zeta(x)$ in distribution and thus

$$
\begin{aligned}
& \sup_{x\in[0,1]} n^{1/2} \left| \left(\widehat{G}(x,x)^{-1/2} - G(x,x)^{-1/2} \right) (\widehat{m}(x) - m(x)) \right| \\
\leq \; & sup_{x\in[0,1]} \left| \widehat{G}(x,x)^{-1/2} - G(x,x)^{-1/2} \right| \sup_{x\in[0,1]} n^{1/2} \left| \widehat{m}(x) - m(x) \right| \\
& \overset{d}{\to} 0 \times \sup_{x\in[0,1]} \left| G^{1/2}(x,x) \zeta(x) \right|,
\end{aligned}
$$

which leads to

$$
n^{1/2} \left(\widehat{G}(x,x)^{-1/2} - G(x,x)^{-1/2} \right) (\widehat{m}(x) - m(x)) \to 0
$$

in distribution as a Banach space valued random sequence. Together with result from Theorem 2 and Theorem 2.3 in Bosq (2000), one obtains the following weak convergence of stochastic processes

$$
n^{1/2} \widehat{G}(\cdot,\cdot)^{-1/2} (\widehat{m}(\cdot) - m(\cdot)) \to \zeta(\cdot).
$$

Applying Continuous Mapping Theorem for Banach space valued random variables (equation (2.11), Bosq (2000)) to $\mathcal{C}[0,1]$-valued random variables $n^{1/2}\widehat{G}(\cdot,\cdot)^{-1/2}(\widehat{m}(\cdot) - m(\cdot))$ and $\zeta(\cdot)$, one obtains the following weak convergence

$$
\begin{aligned}
\sup_{x\in[0,1]} n^{1/2} \left| \widehat{G}(x,x)^{-1/2} (\widehat{m}(x) - m(x)) \right| &\to \sup_{x\in[0,1]} |\zeta(x)|. \\
n^{1/2} \left| \widehat{G}(x,x)^{-1/2} (\widehat{m}(x) - m(x)) \right| &\to |\zeta(x)|, x \in [0,1].
\end{aligned}
$$

The Corollary is thus proved.

S.6 Proof of Theorem 3

According to Theorems 1 and 2, one has

$$
\sup_{x\in[0,1]} \left| n_1^{1/2} G_1(x,x)^{-1/2} (\widehat{m}_1(x) - m_1(x)) - \varsigma_1(x) \right| = o_p(1)
$$

and

$$\sup_{x\in[0,1]}\left|n_2^{1/2}G_2\left(x,x\right)^{-1/2}\left(\widehat{m}_2\left(x\right)-m_2\left(x\right)\right)-\varsigma_2\left(x\right)\right|=o_p\left(1\right),$$

where $\varsigma_i\left(x\right)$, $i=1,2$ are defined similar way as $\varsigma\left(x\right)$ in the proof of Theorem 2, modified for each group accordingly. Thus $\varsigma_i\left(x\right)$ are standardized Gaussian processes with covariance function

$$\mathbb{E}\varsigma_i\left(x\right)\varsigma_i\left(x'\right)=G_i\left(x,x'\right)\left\{G_i\left(x,x\right)G_i\left(x',x'\right)\right\}^{-1/2},x,x'\in[0,1].$$

By the boundness of $G_i\left(x,x'\right)$ and $\lim_{n_1\to\infty}\widehat{r}=r>0$, one obtains

$$\sup_{x\in[0,1]}\left|\frac{n_1^{1/2}\left(\widehat{m}_1-\widehat{m}_2-m_1+m_2\right)\left(x\right)}{\left\{\left(G_1+rG_2\right)\left(x,x\right)\right\}^{1/2}}-\frac{\left(G_1^{1/2}\varsigma_1-rG_2^{1/2}\varsigma_2\right)\left(x,x\right)}{\left\{\left(G_1+rG_2\right)\left(x,x\right)\right\}^{1/2}}\right|=o_p\left(1\right).$$

Define $\varsigma_{12}\left(x\right)=\left(G_1^{1/2}\varsigma_1-rG_2^{1/2}\varsigma_2\right)\left(x,x\right)\left\{\left(G_1+rG_2\right)\left(x,x\right)\right\}^{-1/2}$. Thus

$$\sup_{x\in[0,1]}\left|\frac{n_1^{1/2}\left(\widehat{m}_1-\widehat{m}_2-m_1+m_2\right)\left(x\right)}{\left\{\left(G_1+rG_2\right)\left(x,x\right)\right\}^{1/2}}-\varsigma_{12}\right|=o_p\left(1\right).$$

It can be calculated that $\mathbb{E}\varsigma_{12}\left(x\right)=0$ and

$$\mathbb{E}\varsigma_{12}\left(x\right)\varsigma_{12}\left(x'\right)=\frac{\left(G_1+rG_2\right)\left(x,x'\right)}{\left\{\left(G_1+rG_2\right)\left(x,x\right)\right\}^{1/2}\left\{\left(G_1+rG_2\right)\left(x',x'\right)\right\}^{1/2}},$$

hence $\varsigma_{12}\left(x\right)$ has the same distribution as $\zeta_{12}\left(x\right)$ since both are standardized Gaussian processes with the same covariance function. The Theorem is thus proved by applying Slutsky's Theorem.