STATISTICAL INFERENCE FOR MULTIVARIATE FUNCTIONAL PANEL DATA

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Supplementary Material

This Supplement provides technical lemmas, detailed proofs of the theorems, and additional figures and tables.

S.1 Decomposition

Matrix algebra represents the B-spline estimator $\hat{\boldsymbol{\eta}}_{it}(\cdot)$ in (2.10) as

$$\begin{aligned}\hat{\boldsymbol{\eta}}_{it}(\cdot)^\top &= \left\{\hat{\eta}_{it}^{(1)}(\cdot), \ldots, \hat{\eta}_{it}^{(L)}(\cdot)\right\} \\ &= \left\{B_{1,p}(\cdot), \ldots, B_{J_s+p,p}(\cdot)\right\}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}^\top\boldsymbol{Y}_{it},\end{aligned}$$

where $\boldsymbol{Y}_{it}$ and the design matrix $\boldsymbol{X}$ are respectively

$$(\boldsymbol{Y}_{it})_{N\times L} = (\boldsymbol{Y}_{it1}, \ldots, \boldsymbol{Y}_{itN})^\top = \begin{pmatrix} Y_{it1}^{(1)} & \ldots & Y_{it1}^{(L)} \\ \vdots & \ddots & \vdots \\ Y_{itN}^{(1)} & \ldots & Y_{itN}^{(L)} \end{pmatrix}$$

and

$$\boldsymbol{X}_{N\times(J_s+p)} = \begin{pmatrix} B_{1,p}(1/N) & \dots & B_{J_s+p,p}(1/N) \\ \vdots & \ddots & \vdots \\ B_{1,p}(N/N) & \dots & B_{J_s+p,p}(N/N) \end{pmatrix}.$$

Define the empirical inner product matrix of B-spline basis $\{B_{\ell,p}(\cdot)\}_{\ell=1}^{J_s+p}$ as

$$\boldsymbol{V}_p = \left\{\langle B_{\ell,p}, B_{\ell',p}\rangle_N\right\}_{\ell,\ell'=1}^{J_s+p} = N^{-1}\boldsymbol{X}^\top\boldsymbol{X},$$

where $\langle B_{\ell,p}, B_{\ell',p}\rangle_N = N^{-1}\sum_{j=1}^{N} B_{\ell,p}(j/N)B_{\ell',p}(j/N)$ denotes the empirical inner product of $B_{\ell,p}(\cdot)$ and $B_{\ell',p}(\cdot)$. According to Lemma A.3 in Cao et al. (2012), for some constant $C_p > 0$,

$$\left\|\boldsymbol{V}_p^{-1}\right\|_\infty \leq C_p J_s. \tag{S.1}$$

Define the following matrices,

$$(\boldsymbol{\eta}_{it})_{N\times L} = \{\boldsymbol{\eta}_{it}(1/N), \dots, \boldsymbol{\eta}_{it}(N/N)\}^\top, \quad (\boldsymbol{m})_{N\times L} = \{\boldsymbol{m}(1/N), \dots, \boldsymbol{m}(N/N)\}^\top,$$
$$(\boldsymbol{\xi}_{it})_{N\times L} = \{\boldsymbol{\xi}_{it}(1/N), \dots, \boldsymbol{\xi}_{it}(N/N)\}^\top,$$
$$(\boldsymbol{\varepsilon}_{it})_{N\times L} = \{\boldsymbol{\sigma}_{it}(1/N) \circ \boldsymbol{\varepsilon}_{it1}, \dots, \boldsymbol{\sigma}_{it}(N/N) \circ \boldsymbol{\varepsilon}_{itN}\}^\top,$$

which are matrices of trajectories, vector mean function, centered trajectories and measurement errors on N points of observations respectively. Then

$(\boldsymbol{Y}_{it})_{N\times L} = (\boldsymbol{\eta}_{it})_{N\times L} + (\boldsymbol{\varepsilon}_{it})_{N\times L}$ according to (1.7), and the approximation error $\hat{\boldsymbol{\eta}}_{it}(\cdot) - \boldsymbol{\eta}_{it}(\cdot)$ is decomposed as:

$$\hat{\boldsymbol{\eta}}_{it}(\cdot) - \boldsymbol{\eta}_{it}(\cdot) = \tilde{\boldsymbol{\eta}}_{it}(\cdot) - \boldsymbol{\eta}_{it}(\cdot) + \tilde{\boldsymbol{\varepsilon}}_{it}(\cdot),$$

where

$$\begin{aligned}
\tilde{\boldsymbol{\eta}}_{it}(\cdot)^\top &= N^{-1}\boldsymbol{B}(\cdot)^\top \boldsymbol{V}_p^{-1}\boldsymbol{X}^\top (\boldsymbol{\eta}_{it})_{N\times L} = \tilde{\boldsymbol{m}}(\cdot) + \tilde{\boldsymbol{\xi}}_{it}(\cdot), \\
\tilde{\boldsymbol{m}}(\cdot)^\top &= N^{-1}\boldsymbol{B}(\cdot)^\top \boldsymbol{V}_p^{-1}\boldsymbol{X}^\top (\boldsymbol{m})_{N\times L}, \\
\tilde{\boldsymbol{\xi}}_{it}(\cdot)^\top &= N^{-1}\boldsymbol{B}(\cdot)^\top \boldsymbol{V}_p^{-1}\boldsymbol{X}^\top (\boldsymbol{\xi}_{it})_{N\times L}, \\
\tilde{\boldsymbol{\varepsilon}}_{it}(\cdot)^\top &= N^{-1}\boldsymbol{B}(\cdot)^\top \boldsymbol{V}_p^{-1}\boldsymbol{X}^\top (\boldsymbol{\varepsilon}_{it})_{N\times L},
\end{aligned} \tag{S.2}$$

with $\boldsymbol{B}(\cdot) = \{B_{1,p}(\cdot), \ldots, B_{J_s+p,p}(\cdot)\}^\top$. By (2.9) and (2.10), the approximation error of $\hat{\boldsymbol{m}}(\cdot)$ to $\bar{\boldsymbol{m}}(\cdot)$ is

$$\hat{\boldsymbol{m}}(\cdot) - \bar{\boldsymbol{m}}(\cdot) = (nT)^{-1}\sum_{r(i,t)=1}^{nT}\{\tilde{\boldsymbol{\eta}}_{it}(\cdot) - \boldsymbol{\eta}_{it}(\cdot) + \tilde{\boldsymbol{\varepsilon}}_{it}(\cdot)\}. \tag{S.3}$$

S.2 Ancillary lemmas

For $y > 0$, let $\log^* y = \max\{1, \log y\}$. For $r > 2$, $\alpha > 0$, $L \in \mathbb{N}_+$, define constant $A(r, \alpha, L) = \max\left\{\left(L^{21/2+\alpha}(\log^* L)^2\right)^r, L^{\frac{r(r+2)}{4}}(\log^* L)^{\frac{r(r+1)}{2}}\right\}$.

Lemma S.1 (Theorem 4 of Götze and Zaitsev (2010))**.** *Let* $\boldsymbol{\xi}$ *be a* $\mathbb{R}^L$*-valued* *random vector with* $\mathbb{E}\boldsymbol{\xi} = \mathbf{0}_L$ *and* $\mathbb{E}\left\|\boldsymbol{\xi}\right\|_2^r < \infty$ *for some* $r > 2$*. Let* $\lambda_{\max}\left(\boldsymbol{\Sigma}\right)$ *and* $\lambda_{\min}\left(\boldsymbol{\Sigma}\right)$ *be the maximal and minimal strictly positive eigenvalues of the covariance matrix* $\boldsymbol{\Sigma} = \mathrm{cov}\left(\boldsymbol{\xi}\right)$*, respectively. Then one can construct on a probability space a sequence of independent random vectors* $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots$ *and a corresponding sequence of independent* $N\left(\mathbf{0}, \boldsymbol{\Sigma}\right)$ *random vectors* $\boldsymbol{Z}_1, \boldsymbol{Z}_2, \ldots$ *such that* $\mathcal{L}\left(\boldsymbol{\xi}_i\right) = \mathcal{L}\left(\boldsymbol{\xi}\right)$*, and for all* $x > 0$ *and* $n \in \mathbb{N}_+$*, the following inequality is valid:*

$$\mathbb{P}\left\{\max_{1\leq\tau\leq n}\left\|\sum_{i=1}^{\tau}\boldsymbol{\xi}_i - \sum_{i=1}^{\tau}\boldsymbol{Z}_i\right\|_2 > x\right\} \leq cA\left\{\lambda_{\max}\left(\boldsymbol{\Sigma}\right)/\lambda_{\min}\left(\boldsymbol{\Sigma}\right)\right\}^{r/2} n\mathbb{E}\left\|\boldsymbol{\xi}\right\|_2^r / x^r,$$

where $A = A(r, \alpha, L)$ *and* c *is a positive quantity depending on* r *and* α *only with* $\alpha > 0$*.*

Lemma S.2 (Theorem 7.5 of Billingsley (1999))**.** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and let* X *map* Ω *into* $C[0,1]$*:* $X(\omega)$ *is an element of* $C[0,1]$ *with value* $X_t(\omega) = X(t, \omega)$ *at* t*. For* $F \in C[0,1]$*, denote* $\omega(F,h) = \sup_{x,x'\in[0,1],|x-x'|\leq h}|F(x') - F(x)|$ *as the modulus of continuity. Suppose that* $X, X^1, X^2, \ldots$ *are random functions. If* $\left(X_{t_1}^n, \ldots, X_{t_d}^n\right) \to_D \left(X_{t_1}, \ldots, X_{t_d}\right)$ *holds for all* $t_1, \ldots, t_d$*, and if*

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbb{P}\left[\omega\left(X^n, \delta\right) \geq \epsilon\right] = 0 \tag{S.4}$$

for each positive ϵ*, then* $X^n \to_D X$.

Lemma S.3 (Theorem 29.16 of Davidson (1994)). *Define* $(\mathcal{C}[0,1])^L$ *as the space of L-vectors of continuous functions on* $[0,1]$, $(\mathcal{D}[0,1])^L$ *as the space of L-vectors of cadlag functions on [0,1]. Let* $\boldsymbol{X}_n \in (\mathcal{D}[0,1])^L$ *be an L-vector of random functions. Then* $\boldsymbol{X}_n \to_D \boldsymbol{X}$*, where* $\mathbb{P}\left\{\boldsymbol{X} \in (\mathcal{C}[0,1])^L\right\} = 1$*, iff* $\boldsymbol{\lambda}^\top \boldsymbol{X}_n \to_D \boldsymbol{\lambda}^\top \boldsymbol{X}$ *for every fixed* $\boldsymbol{\lambda} \in \mathbb{R}^L$ *with* $\boldsymbol{\lambda}^\top \boldsymbol{\lambda} = 1$.

Lemma S.4 (Lemma A.14 of Wang et al. (2020)). *For* $n > 2$*,* $a > 2$*,* $W_i \sim N\left(0, \sigma_i^2\right), \sigma_i > 0, i = 1, \ldots, n$

$$\mathbb{P}\left(\max_{1\le i\le n} |W_i/\sigma_i| > a\sqrt{\log n}\right) < \sqrt{\frac{2}{\pi}} n^{1-a^2/2}. \tag{S.5}$$

As $n \to \infty, \left(\max_{1\le i\le n} |W_i|\right) / \left(\max_{1\le i\le n} \sigma_i\right) \le \max_{1\le i\le n} |W_i/\sigma_i| = \mathcal{O}_{a.s.}\left(\sqrt{\log n}\right)$.

Lemma S.5. *Assumption (A5) holds under Assumptions (A3) and (A5').*

Proof. Under Assumption (A5'), $\mathbb{E}\left\|\boldsymbol{\zeta}_{itk}\right\|_2^{r_1} < \infty$, $r_1 > 4 + 2\omega$ with some $\omega > 0$, thus there exists some $\beta_0 \in (0, 1/2)$, such that $r_1 > (2+\omega)/\beta_0$. Lemma S.1 entails that for each $1 \le k \le k_{nT}$, one can construct $\left\{\tilde{\boldsymbol{\zeta}}_{itk}\right\}_{r(i,t)=1-nI_{nT}}^{nT}$ equal in distribution to $\left\{\boldsymbol{\zeta}_{itk}\right\}_{r(i,t)=1-nI_{nT}}^{nT}$ and $N(\mathbf{0}_L, \boldsymbol{\Sigma}_{k,\zeta})$ random vectors

$\{\boldsymbol{Z}_{itk,\zeta}\}_{r(i,t)=1-nI_{nT}}^{nT}$ on a new probability space $\left(\tilde{\Omega}_{k,\zeta}, \tilde{\mathcal{A}}_{k,\zeta}, \tilde{\mathbb{P}}_{k,\zeta}\right)$, such that,

$$\tilde{\mathbb{P}}_{k,\zeta}\left\{\max_{1-nI_{nT}\leq\tau\leq nT}\left\|\sum_{r(i,t)=1-nI_{nT}}^{\tau}\left(\tilde{\boldsymbol{\zeta}}_{itk}-\boldsymbol{Z}_{itk,\zeta}\right)\right\|_2 > \{n(I_{nT}+T)\}^{\beta_0}\right\}$$
$$< c_1 A_1 c_\lambda^{r_1/2}\mathbb{E}\left\|\boldsymbol{\zeta}_{11,k}\right\|^{r_1}\{n(I_{nT}+T)\}^{1-r_1\beta_0},$$

where c_1 and A_1 do not vary with k. Likewise, under Assumption (A5'), $\mathbb{E}\left\|\boldsymbol{\varepsilon}_{itj}\right\|_2^{r_2} < \infty$ with $r_2 > (2+\theta)/\beta_2$. For each $1 \leq r(i,t) \leq nT$, one can construct $\{\tilde{\boldsymbol{\varepsilon}}_{itj}\}_{j=1}^N$ equal in distribution to $\{\boldsymbol{\varepsilon}_{itj}\}_{j=1}^N$, and $N(\mathbf{0}_L, \boldsymbol{\Sigma}_{it,\varepsilon})$ random vectors $\{\boldsymbol{Z}_{itj,\varepsilon}\}_{j=1}^N$ on a new probability space $\left(\tilde{\Omega}_{it,\varepsilon}, \tilde{\mathcal{A}}_{it,\varepsilon}, \tilde{\mathbb{P}}_{it,\varepsilon}\right)$ such that

$$\tilde{\mathbb{P}}_{it,\varepsilon}\left\{\max_{1\leq\tau\leq N}\left\|\sum_{j=1}^{\tau}\tilde{\boldsymbol{\varepsilon}}_{itj}-\sum_{j=1}^{\tau}\boldsymbol{Z}_{itj,\varepsilon}\right\|_2 > N^{\beta_2}\right\} < c_2 A_2 c_\lambda^{r_2/2}\mathbb{E}\left\|\boldsymbol{\varepsilon}_{it,1}\right\|^{r_2} N^{1-r_2\beta_2},$$

where c_2 and A_2 do not vary with (i,t).

Since Assumption (A5') stipulates the independence of $\{\boldsymbol{\zeta}_{itk}\}_{r(i,t)=1-nI_{nT},k=1}^{nT,\infty}$ and $\{\boldsymbol{\varepsilon}_{itj}\}_{r(i,t)=1,j=1}^{nT,N}$, the independence is automatically preserved for $\left\{\tilde{\boldsymbol{\zeta}}_{itk}\right\}_{r(i,t)=1-nI_{nT},k=1}^{nT,k_{nT}}$ and $\{\tilde{\boldsymbol{\varepsilon}}_{itj}\}_{r(i,t)=1,j=1}^{nT,N}$ if their new probability space $\left(\tilde{\Omega}_{k,\zeta}, \tilde{\mathcal{A}}_{k,\zeta}, \tilde{\mathbb{P}}_{k,\zeta}\right)$, $k \geq 1$ and $\left(\tilde{\Omega}_{it,\varepsilon}, \tilde{\mathcal{A}}_{it,\varepsilon}, \tilde{\mathbb{P}}_{it,\varepsilon}\right)$, $r(i,t) \geq 1$ are all independently embedded into a product probability space

$$\left(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbb{P}}\right) \equiv \left(\left(\bigotimes_{k=1}^{\infty}\tilde{\Omega}_{k,\zeta}\right)\bigotimes\left(\bigotimes_{r(i,t)=1}^{\infty}\tilde{\Omega}_{it,\varepsilon}\right), \left(\bigotimes_{k=1}^{\infty}\tilde{\mathcal{A}}_{k,\zeta}\right)\bigotimes\left(\bigotimes_{r(i,t)=1}^{\infty}\tilde{\mathcal{A}}_{it,\varepsilon}\right), \tilde{\mathbb{P}}\right)$$

according to Ionescu-Tulcea Theorem (Theorem 14.32 in Klenke (2014)). This independent embedding also ensures that all Gaussian random vectors $\{\boldsymbol{Z}_{itk,\zeta}\}_{r(i,t)=1-nI_{nT},k=1}^{nT,k_{nT}}$, $\{\boldsymbol{Z}_{itj,\varepsilon}\}_{r(i,t)=1,j=1}^{nT,N}$ remain independent in the new product probability space, as required in the Assumption (A5).

In what follows, with some abuse of notations, we will not distinguish $\boldsymbol{\zeta}_{itk}$, $\boldsymbol{\varepsilon}_{itj}$ on the original probability space from $\tilde{\boldsymbol{\zeta}}_{itk}$, $\tilde{\boldsymbol{\varepsilon}}_{itj}$ on the above product probability space, nor the original probability measure $\mathbb{P}$ from $\tilde{\mathbb{P}}$ on the product space.

Since $\sup_{k\geq 1}\mathbb{E}\left\|\boldsymbol{\zeta}_{11,k}\right\|_2^{r_1} < \infty$ by Assumption (A5'), there exists a common $c > 0$, such that

$$\max_{1\leq k\leq k_{nT}} \mathbb{P}\left\{\max_{1-nI_{nT}\leq \tau\leq nT}\left\|\sum_{r(i,t)=1-nI_{nT}}^{\tau}(\boldsymbol{\zeta}_{itk}-\boldsymbol{Z}_{itk,\zeta})\right\|_2 > \{n(I_{nT}+T)\}^{\beta_0}\right\}$$
$$< c\{n(I_{nT}+T)\}^{1-r_1\beta_0}.$$

Noting that $I_{nT} \asymp \log(nT)$ and $\log^{-1}(nT)\log(1+I_{nT}/T) \to 0$ as $nT \to \infty$ by the definition, $\log^{-1}(nT)\log(1+I_{nT}/T) < \epsilon < 1/2-\beta_0$ is solvable for ϵ. Denote $\beta_1 = \beta_0+\epsilon$, then $(nT)^{\beta_1} > \{n(I_{nT}+T)\}^{\beta_0}$, $0 < \beta_1 < 1/2$. Since $r_1 > (2+\omega)/\beta_0$, one can let $\gamma_1 = r_1\beta_0 - 1 - \omega > 1$, then there exists a

$C_1 > 0$ such that

$$\mathbb{P}\left\{\max_{1\le k\le k_{nT}}\max_{1-nI_{nT}\le\tau\le nT}\left\|\sum_{r(i,t)=1-nI_{nT}}^{\tau}(\boldsymbol{\zeta}_{itk}-\boldsymbol{Z}_{itk,\zeta})\right\|_2 > (nT)^{\beta_1}\right\}$$
$$< k_{nT}\ c\{n(I_{nT}+T)\}^{1-r_1\beta_0} < C_1\,(nT)^{-\gamma_1}.$$

Since $\sup_{r(i,t)\ge 1}\mathbb{E}\left\|\boldsymbol{\varepsilon}_{it,1}\right\|_2^{r_1} < \infty$, there exists a constant c' such that

$$\max_{1\le r(i,t)\le nT}\mathbb{P}\left\{\max_{1\le\tau\le N}\left\|\sum_{j=1}^{\tau}\boldsymbol{\varepsilon}_{itj}-\sum_{j=1}^{\tau}\boldsymbol{Z}_{itj,\varepsilon}\right\|_2 > N^{\beta_2}\right\} < c'N^{1-r_2\beta_2}.$$

Denote $\gamma_2 = r_2\beta_2 - 1 - \theta$, then $\gamma_2 > 1$ since $r_2 > (2+\theta)/\beta_2$ under Assumption (A6). Assumption (A3) states that $nT = \mathcal{O}\left(N^{\theta}\right)$, so there is a $C_2 > 0$ such that

$$\mathbb{P}\left\{\max_{1\le r(i,t)\le nT}\max_{1\le\tau\le N}\left\|\sum_{j=1}^{\tau}\boldsymbol{\varepsilon}_{itj}-\sum_{j=1}^{\tau}\boldsymbol{Z}_{itj,\varepsilon}\right\|_2 > N^{\beta_2}\right\} < c'\times(nT)\times N^{1-r_2\beta_2}$$
$$< C_2N^{-\gamma_2}.$$

□

Lemma S.6. *Under Assumption (A5), as $T\to\infty$, for $N\left(\mathbf{0}_L,\boldsymbol{\Sigma}_{k,\xi}\right)$ random vectors $\{\boldsymbol{Z}_{itk,\xi}\}_{r(i,t)=1}^{nT}$ defined by $\boldsymbol{Z}_{itk,\xi} = \sum_{t'=0}^{\infty}\boldsymbol{a}_{t'k}\circ\boldsymbol{Z}_{i,t-t',k,\zeta}$ with $\boldsymbol{\Sigma}_{k,\xi} = \boldsymbol{\Sigma}_{k,\zeta}\circ\sum_{t=0}^{\infty}\boldsymbol{a}_{tk}\boldsymbol{a}_{tk}^{\top}$ and $\{\boldsymbol{Z}_{itk,\zeta}\}$ in Assumption (A5), $1\le k\le k_{nT}$, there* *exist constants* $C_3,\ C_4 \in (0,+\infty),\ \gamma_3 \in (0,\infty),$ *such that*

$$\mathbb{P}\left\{\max_{1\le k\le k_{nT}}\max_{1\le\tau\le nT}\left\|\sum_{r(i,t)=1}^{\tau}\left(\boldsymbol{\xi}_{itk}-\boldsymbol{Z}_{itk,\xi}\right)\right\|_2 > C_3\,(nT)^{\beta_1}\right\} < C_4\,(nT)^{-\gamma_3}.$$

Proof. One decomposes the difference of partial sum as follows:

$$\max_{1\le k\le k_{nT}}\max_{1\le\tau\le nT}\left\|\sum_{r(i,t)=1}^{\tau}\left(\boldsymbol{\xi}_{itk}-\boldsymbol{Z}_{itk,\xi}\right)\right\|_2 \le D_1 + D_2 + D_3, \tag{S.6}$$

where

$$D_1 = \max_{1\le k\le k_{nT}}\max_{1\le\tau\le nT}\left\|\sum_{r(i,t)=1}^{\tau}\left(\boldsymbol{\xi}_{itk}-\sum_{t'=0}^{I_{nT}}\boldsymbol{a}_{t'k}\circ\boldsymbol{\zeta}_{i,t-t',k}\right)\right\|_2,$$

$$D_2 = \max_{1\le k\le k_{nT}}\max_{1\le\tau\le nT}\left\|\sum_{r(i,t)=1}^{\tau}\sum_{t'=0}^{I_{nT}}\boldsymbol{a}_{t'k}\circ\left(\boldsymbol{\zeta}_{i,t-t',k}-\boldsymbol{Z}_{i,t-t',k,\zeta}\right)\right\|_2,$$

$$D_3 = \max_{1\le k\le k_{nT}}\max_{1\le\tau\le nT}\left\|\sum_{r(i,t)=1}^{\tau}\sum_{t'=I_{nT}+1}^{\infty}\boldsymbol{a}_{t'k}\circ\boldsymbol{Z}_{i,t-t',k,\zeta}\right\|_2.$$

Since $\|\boldsymbol{a}_{tk}\|_\infty < C_a\rho_a^t$, there exists a constant $M > 0$ such that $\sum_{t=0}^{\infty}\|\boldsymbol{a}_{tk}\|_\infty < M$, $k \ge 1$, then

$$
\begin{aligned}
&\mathbb{P}\left\{D_2 > 2M\left(nT\right)^{\beta_1}\right\} \\
=&\mathbb{P}\left\{\max_{1\le k\le k_{nT}}\max_{1\le\tau\le nT}\left\|\sum_{r(i,t)=1}^{\tau}\sum_{t'=0}^{I_{nT}}\boldsymbol{a}_{t'k}\circ\left(\boldsymbol{\zeta}_{i,t-t',k}-\boldsymbol{Z}_{i,t-t',k,\zeta}\right)\right\|_2 > 2M\left(nT\right)^{\beta_1}\right\} \\
\le&\mathbb{P}\left\{\max_{1\le k\le k_{nT}}\max_{1\le\tau\le nT}\sum_{t'=0}^{I_{nT}}\left\|\boldsymbol{a}_{t'k}\right\|_\infty\left\|\sum_{r(i,t)=1}^{\tau}\left(\boldsymbol{\zeta}_{i,t-t',k}-\boldsymbol{Z}_{i,t-t',k,\zeta}\right)\right\|_2 > 2M\left(nT\right)^{\beta_1}\right\} \\
\le&\mathbb{P}\left\{M\max_{1\le k\le k_{nT}}\max_{1\le\tau\le nT}\max_{0\le t'\le I_{nT}}\left\|\sum_{r(i,t)=1}^{\tau}\left(\boldsymbol{\zeta}_{i,t-t',k}-\boldsymbol{Z}_{i,t-t',k,\zeta}\right)\right\|_2 > 2M\left(nT\right)^{\beta_1}\right\} \\
\le&\mathbb{P}\left\{2M\max_{1\le k\le k_{nT}}\max_{1-nI_{nT}\le\tau\le nT}\left\|\sum_{r(i,t)=1-nI_{nT}}^{\tau}\left(\boldsymbol{\zeta}_{itk}-\boldsymbol{Z}_{itk,\zeta}\right)\right\|_2 > 2M\left(nT\right)^{\beta_1}\right\} < C_1\left(nT\right)^{-\gamma_1},
\end{aligned}
$$

in which the last inequality is entailed by Assumption (A5).

Note that

$$
\begin{aligned}
D_1 &= \max_{1\le k\le k_{nT}}\max_{1\le\tau\le nT}\left\|\sum_{r(i,t)=1}^{\tau}\sum_{t'=I_{nT}+1}^{\infty}\boldsymbol{a}_{t'k}\circ\boldsymbol{\zeta}_{i,t-t',k}\right\|_2 \\
&\le nT\max_{1\le k\le k_{nT}}\max_{1\le r(i,t)\le nT}\left\|\sum_{t'=I_{nT}+1}^{\infty}\boldsymbol{a}_{t'k}\circ\boldsymbol{\zeta}_{i,t-t',k}\right\|_2 .
\end{aligned}
$$

With $I_{nT} > -10\log\left(nT\right)/\log\rho_a$, then $\rho_a^{I_{nT}} < \left(nT\right)^{-10}$, $\left\|\boldsymbol{a}_{t'k}\right\|_\infty < C_a\left(nT\right)^{-10}\rho_a^{t'-I_{nT}}$

when $t' > I_{nT}$, then

$$\left\| \sum_{t'=I_{nT}+1}^{\infty} \boldsymbol{a}_{t'k} \circ \boldsymbol{\zeta}_{i,t-t',k} \right\|_2 \leq \sum_{t'=I_{nT}+1}^{\infty} C_a (nT)^{-10} \rho_a^{t'-I_{nT}} \left\| \boldsymbol{\zeta}_{i,t-t',k} \right\|_2$$
$$= C_a (nT)^{-10} \sum_{t'=1}^{\infty} \rho_a^{t'} \left\| \boldsymbol{\zeta}_{i,t-I_{nT}-t',k} \right\|_2 .$$

Denote $W_{itk} = \sum_{t'=1}^{\infty} \rho_a^{t'} \left\| \boldsymbol{\zeta}_{i,t-I_{nT}-t',k} \right\|_2$, then

$$D_1 \leq C_a (nT)^{-9} \max_{1\leq k\leq k_{nT}} \max_{1\leq r(i,t)\leq n} W_{itk}.$$

By noticing that $\sup_k \mathbb{E} \left\| \boldsymbol{\zeta}_{it,k} \right\|_2^{r_0} < \infty$, one has

$$\max_{1\leq k\leq k_{nT}} (\mathbb{E} W_{itk}^{r_0})^{1/r_0} \leq \max_{1\leq k\leq k_{nT}} (\mathbb{E} \| \boldsymbol{\zeta}_{itk} \|_2^{r_0})^{1/r_0} \sum_{t'=1}^{\infty} \rho_a^{t'} < \infty.$$

Therefore, $\max_{1\leq k\leq k_{nT}} \mathbb{E} W_{itk}^{r_0} < K$ for some $K > 0$. Note that $k_{nT} = \mathcal{O}\left\{(nT)^{\omega}\right\}$ in Assumption (A5), thus

$$\mathbb{P}\left\{D_1 > M (nT)^{\beta_1}\right\} \leq \mathbb{P}\left\{C_a (nT)^{-9} \max_{1\leq k\leq k_{nT}} \max_{1\leq r(i,t)\leq nT} W_{itk} > M (nT)^{\beta_1}\right\}$$
$$< k_{nT} \times nT \times \frac{C_a^{r_0} K}{M^{r_0}} (nT)^{-(\beta_1+9)r_0} < \frac{C_a^{r_0} K'}{M^{r_0}} (nT)^{-(\beta_1+9)r_0+1+\omega} .$$

Finally, define random vectors $\boldsymbol{W}'_{itk} = \sum_{t'=I_{nT}+1}^{\infty} \boldsymbol{a}_{t'k} \circ \boldsymbol{Z}_{i,t-t',k,\zeta}$, then

$$D_3 = \max_{1\leq k\leq k_{nT}} \max_{1\leq \tau\leq nT} \left\| \sum_{r(i,t)=1}^{\tau} \boldsymbol{W}'_{itk} \right\|_2 \leq nT \max_{1\leq k\leq k_{nT}} \max_{1\leq r(i,t)\leq nT} \| \boldsymbol{W}'_{itk} \|_2 .$$

Noting that $\mathbb{E}\left\|\boldsymbol{W}'_{itk}\right\|_2^2 = \sum_{t=I_{nT}+1}^{\infty}\left\|\boldsymbol{a}_{tk}\right\|_2^2 < C(nT)^{-20}$ holds for $k \in \mathbb{N}_+$ uniformly, one has

$$\mathbb{P}\left\{D_3 > M\left(nT\right)^{\beta_1}\right\} \leq \mathbb{P}\left\{\max_{1\leq k\leq k_{nT}}\max_{1\leq r(i,t)\leq nT}\left\|\boldsymbol{W}'_{itk}\right\|_2 > M\left(nT\right)^{\beta_1-1}\right\}$$
$$< k_{nT} \times nT \times \frac{\max_{1\leq k\leq k_{nT}}\mathbb{E}\left\|\boldsymbol{W}'_{itk}\right\|_2^2}{M^2\left(nT\right)^{2(\beta_1-1)}} < \frac{C}{M^2}\left(nT\right)^{-17-2\beta_1+\omega}.$$

By decomposition (S.6), one obtains that

$$\mathbb{P}\left\{\max_{1\leq k\leq k_{nT}}\max_{1\leq \tau\leq nT}\left\|\sum_{r(i,t)=1}^{\tau}\left(\boldsymbol{\xi}_{itk} - \boldsymbol{Z}_{itk,\xi}\right)\right\|_2 > 4M\left(nT\right)^{\beta_1}\right\}$$
$$\leq \mathbb{P}\left\{D_2 > 2M\left(nT\right)^{\beta_1}\right\} + \mathbb{P}\left\{D_1 > M\left(nT\right)^{\beta_1}\right\} + \mathbb{P}\left\{D_3 > M\left(nT\right)^{\beta_1}\right\}$$
$$< C_1\left(nT\right)^{-\gamma_3} + \frac{C_a^{r_0}K'}{M^{r_0}}\left(nT\right)^{-(\beta_1+9)r_0+1+\omega} + \frac{C}{M^2}\left(nT\right)^{-17-2\beta_1+\omega},$$

in which the first term is dominating. Denote $C_3 = 4M$, then there exists constant C_4 such that

$$\mathbb{P}\left\{\max_{1\leq k\leq k_{nT}}\max_{1\leq \tau\leq nT}\left\|\sum_{r(i,t)=1}^{\tau}\left(\boldsymbol{\xi}_{itk} - \boldsymbol{Z}_{itk,\xi}\right)\right\|_2 > C_3\left(nT\right)^{\beta_1}\right\} < C_4\left(nT\right)^{-\gamma_3}.$$

□

Lemma S.7. *Under Assumptions (A2) and (A6), as $T \to \infty$,*

$$\max_{1\le\ell\le J_s+p}\left\|(nTN)^{-1}\sum_{j=1}^{N}\sum_{r(i,t)=1}^{nT} B_{\ell,p}\left(j/N\right)\boldsymbol{\sigma}_{it}(j/N)\circ\boldsymbol{Z}_{itj,\varepsilon}\right\|_{\infty}$$
$$=\mathcal{O}_{a.s.}\left\{(nT)^{-1/2}N^{-1/2}J_s^{-1/2}\log^{1/2}N\right\}.$$

Proof. Denote random vector $\bar{\boldsymbol{Z}}_{\cdot\cdot j,\varepsilon}=\left(\bar{Z}^{(1)}_{\cdot\cdot j,\varepsilon},\ldots,\bar{Z}^{(L)}_{\cdot\cdot j,\varepsilon}\right)^{\top}=(nT)^{-1}\sum_{r(i,t)=1}^{nT}\boldsymbol{\sigma}_{it}(j/N)\circ$ $\boldsymbol{Z}_{itj,\varepsilon}$, and note that

$$\begin{aligned}\mathbb{E}\left\{N^{-1}\sum_{j=1}^{N}B_{\ell,p}\left(j/N\right)\bar{Z}^{(l)}_{\cdot\cdot j,\varepsilon}\right\}^{2}=&N^{-2}\sum_{j=1}^{N}B^{2}_{\ell,p}\left(\frac{j}{N}\right)\mathbb{E}\bar{Z}^{(l)}_{\cdot\cdot j,\varepsilon}{}^{2}\\ &\le(nT)^{-1}N^{-1}\left\|B_{\ell,p}\right\|^{2}_{2,N}\max_{1\le r(i,t)\le nT}\sup_{x\in[0,1]}\left\|\boldsymbol{\sigma}_{it}(x)\right\|^{2}_{\infty}\\ &\asymp(nT)^{-1}N^{-1}J_s^{-1}\end{aligned}$$

uniformly for $1\le\ell\le J_s+p$ and $1\le l\le L$, with $\left\|B_{\ell,p}\right\|_{2,N}=\langle B_{\ell,p},B_{\ell,p}\rangle_N^{1/2}$.

It follows from Lemma S.4 that

$$\begin{aligned}\max_{1\le\ell\le J_s+p}\left\|N^{-1}\sum_{j=1}^{N}B_{\ell,p}\left(j/N\right)\bar{\boldsymbol{Z}}_{\cdot\cdot j,\varepsilon}\right\|_{\infty}&=\max_{1\le\ell\le J_s+p}\max_{1\le l\le L}\left|N^{-1}\sum_{j=1}^{N}B_{\ell,p}\left(j/N\right)\bar{Z}^{(l)}_{\cdot\cdot j,\varepsilon}\right|\\ &=\mathcal{O}_{a.s.}\left\{(nT)^{-1/2}N^{-1/2}J_s^{-1/2}\log^{1/2}\left(J_s+p\right)\right\}\\ &=\mathcal{O}_{a.s.}\left\{(nT)^{-1/2}N^{-1/2}J_s^{-1/2}\log^{1/2}\left(N\right)\right\},\end{aligned}$$

where the last step follows from Assumption (A6) on the order of J_s relative to N. □

Lemma S.8. *Under Assumptions (A2), (A5) and (A6), one has*

$$\sup_{x\in[0,1]}\left\|(nT)^{-1}\sum_{r(i,t)=1}^{nT}\tilde{\boldsymbol{\varepsilon}}_{it}\left(x\right)\right\|_{\infty}=\mathcal{O}_{a.s.}\left\{(nT)^{-1/2}N^{-1/2}J_s^{1/2}\log^{1/2}N+N^{\beta_2-1}J_s\right\}.$$

Proof. According to Assumption (A5), one has

$$\max_{1\leq r(i,t)\leq nT}\max_{1\leq\tau\leq N}\left\|N^{-1}\sum_{j=1}^{\tau}(\boldsymbol{\varepsilon}_{itj}-\boldsymbol{Z}_{itj,\varepsilon})\right\|_2=\mathcal{O}_{a.s.}(N^{\beta_2-1}).$$

Next, the B spline basis satisfies

$$\left|B_{\ell,p}\left(\frac{j}{N}\right)-B_{\ell,p}\left(\frac{j+1}{N}\right)\right|\leq N^{-1}\left\|B_{\ell,p}\right\|_{0,1}\leq CJ_sN^{-1}$$

uniformly over $1\leq j\leq N$ and $1\leq\ell\leq J_s+p$, and the stardard deviation functions satisfy

$$\left\|\boldsymbol{\sigma}_{it}\left(\frac{j}{N}\right)-\boldsymbol{\sigma}_{it}\left(\frac{j+1}{N}\right)\right\|_{\infty}\leq N^{-\nu}\left\|\boldsymbol{\sigma}_{it}\right\|_{0,\nu}\leq CJ_sN^{-1}$$

uniformly over $1\leq j\leq N, 1\leq r\left(i,t\right)\leq nT$, in which the first inequality is entailed by Assumption (A2), and the second inequality holds since $J_sN^{-1}=N^{\gamma-1}d_N\gg N^{-\nu}$ by Assumption (A6). Note that $B_{\ell,p}\left(\cdot\right)$ and $\boldsymbol{\sigma}_{it}\left(\cdot\right)$ are bounded on $[0,1]$ uniformly for $1\leq\ell\leq J_s+p$ and $1\leq r(i,t)\leq nT$

respectively, then

$$\begin{aligned}
&\left\| B_{\ell,p}\left(\frac{j}{N}\right)\boldsymbol{\sigma}_{it}\left(\frac{j}{N}\right) - B_{\ell,p}\left(\frac{j+1}{N}\right)\boldsymbol{\sigma}_{it}\left(\frac{j+1}{N}\right)\right\|_{\infty} \\
=&\left\|\left\{B_{\ell,p}\left(\frac{j}{N}\right) - B_{\ell,p}\left(\frac{j+1}{N}\right) + B_{\ell,p}\left(\frac{j+1}{N}\right)\right\}\boldsymbol{\sigma}_{it}\left(\frac{j}{N}\right)\right. \\
&\qquad\qquad \left. -B_{\ell,p}\left(\frac{j+1}{N}\right)\boldsymbol{\sigma}_{it}\left(\frac{j+1}{N}\right)\right\|_{\infty} \\
\leq&\left\|\left\{B_{\ell,p}\left(\frac{j}{N}\right) - B_{\ell,p}\left(\frac{j+1}{N}\right)\right\}\boldsymbol{\sigma}_{it}\left(\frac{j}{N}\right)\right\|_{\infty} \\
&+\left\|\left\{\boldsymbol{\sigma}_{it}\left(\frac{j}{N}\right) - \boldsymbol{\sigma}_{it}\left(\frac{j+1}{N}\right)\right\}B_{\ell,p}\left(\frac{j+1}{N}\right)\right\|_{\infty} \leq CJ_sN^{-1}.
\end{aligned}$$

Since the support of $B_{\ell,p}(\cdot)$ has length at most $p/(J_s+1)$, one obtains that for $1 \leq \ell \leq J_s + p$,

$$\sum_{j=1}^{N-1}\left\| B_{\ell,p}\left(\frac{j}{N}\right)\boldsymbol{\sigma}_{it}\left(\frac{j}{N}\right) - B_{\ell,p}\left(\frac{j+1}{N}\right)\boldsymbol{\sigma}_{it}\left(\frac{j+1}{N}\right)\right\|_{\infty} \leq C.$$

Hence,

$$\max_{1\le\ell\le J_s+p}\left\|(nTN)^{-1}\sum_{r(i,t)=1}^{nT}\sum_{j=1}^{N}B_{\ell,p}\left(j/N\right)\boldsymbol{\sigma}_{it}\left(j/N\right)\circ\left(\boldsymbol{\varepsilon}_{itj}-\boldsymbol{Z}_{itj,\varepsilon}\right)\right\|_{\infty}$$
$$=\max_{1\le\ell\le J_s+p}\left\|(nT)^{-1}\sum_{r(i,t)=1}^{nT}\sum_{j=1}^{N-1}\left[\left\{B_{\ell,p}\left(\frac{j}{N}\right)\boldsymbol{\sigma}_{it}\left(\frac{j}{N}\right)-B_{\ell,p}\left(\frac{j+1}{N}\right)\boldsymbol{\sigma}_{it}\left(\frac{j+1}{N}\right)\right\}\right.\right.$$
$$\left.\left.\circ\left\{N^{-1}\sum_{s=1}^{j}\left(\boldsymbol{\varepsilon}_{its}-\boldsymbol{Z}_{its,\varepsilon}\right)\right\}\right]+(nT)^{-1}\sum_{r(i,t)=1}^{nT}\left\{B_{\ell,p}\left(1\right)\boldsymbol{\sigma}_{it}\left(1\right)\circ N^{-1}\sum_{j=1}^{N}\left(\boldsymbol{\varepsilon}_{itj}-\boldsymbol{Z}_{itj,\varepsilon}\right)\right\}\right\|_{\infty}$$
$$\le\max_{1\le r(i,t)\le nT}\left\{\max_{1\le j\le N}\left\|N^{-1}\sum_{s=1}^{j}\left(\boldsymbol{\varepsilon}_{its}-\boldsymbol{Z}_{its,\varepsilon}\right)\right\|_{\infty}\right\}$$
$$\times\left\{\max_{1\le\ell\le J_s+p}\sum_{j=1}^{N-1}\left\|B_{\ell,p}\left(\frac{j}{N}\right)\boldsymbol{\sigma}_{it}\left(\frac{j}{N}\right)-B_{\ell,p}\left(\frac{j+1}{N}\right)\boldsymbol{\sigma}_{it}\left(\frac{j+1}{N}\right)\right\|_{\infty}+\left\|B_{\ell,p}(1)\boldsymbol{\sigma}_{it}(1)\right\|_{\infty}\right\}$$
$$=\mathcal{O}_{a.s.}\left(N^{\beta_2-1}\right).$$

The above inequality and Lemma (S.7) together imply that

$$\max_{1\le\ell\le J_s+p}\left\|(nTN)^{-1}\sum_{r(i,t)=1}^{nT}\sum_{j=1}^{N}B_{\ell,p}\left(j/N\right)\boldsymbol{\sigma}_{it}\left(j/N\right)\circ\boldsymbol{\varepsilon}_{itj}\right\|_{\infty}$$
$$=\mathcal{O}_{a.s.}\left\{(nT)^{-1/2}N^{-1/2}J_s^{-1/2}\log^{1/2}N+N^{\beta_2-1}\right\}. \tag{S.7}$$

Recall that $\tilde{\boldsymbol{\varepsilon}}_{it}(\cdot)^\top = N^{-1}\boldsymbol{B}(\cdot)^\top \boldsymbol{V}_p^{-1}\boldsymbol{X}^\top\boldsymbol{\varepsilon}_{it}$, and note that

$$\left\{(nTN)^{-1}\boldsymbol{X}^\top \sum_{r(i,t)=1}^{nT}\boldsymbol{\varepsilon}_{it}\right\}_{(J_s+p)\times L}$$
$$= \left\{(nTN)^{-1}\sum_{r(i,t)=1}^{nT}\sum_{j=1}^{N}B_{\ell,p}(j/N)\,\sigma_{it}^{(l)}(j/N)\,\varepsilon_{itj}^{(l)}\right\}_{\ell=1,l=1}^{J_s+p,L},$$

Thus

$$\sup_{x\in[0,1]}\left\|(nT)^{-1}\sum_{r(i,t)=1}^{nT}\tilde{\boldsymbol{\varepsilon}}_{it}(x)\right\|_\infty$$
$$=\sup_{x\in[0,1]}\left\|B(x)^\top\boldsymbol{V}_p^{-1}\left\{(nTN)^{-1}\boldsymbol{X}^\top\sum_{r(i,t)=1}^{nT}\boldsymbol{\varepsilon}_{it}\right\}\right\|_\infty$$
$$\asymp\max_{1\le l\le L}\left\|\boldsymbol{V}_p^{-1}\left\{(nTN)^{-1}\sum_{r(i,t)=1}^{nT}\sum_{j=1}^{N}B_{\ell,p}(j/N)\,\sigma_{it}^{(l)}(j/N)\,\varepsilon_{itj}^{(l)}\right\}_{\ell=1}^{J_s+p}\right\|_\infty$$
$$\le C_pJ_s\max_{1\le l\le L}\left\|\left\{(nTN)^{-1}\sum_{r(i,t)=1}^{nT}\sum_{j=1}^{N}B_{\ell,p}(j/N)\,\sigma_{it}^{(l)}(j/N)\,\varepsilon_{itj}^{(l)}\right\}_{\ell=1}^{J_s+p}\right\|_\infty$$
$$=\mathcal{O}_{a.s.}\left\{(nT)^{-1/2}N^{-1/2}J_s^{1/2}\log^{1/2}N+J_sN^{\beta_2-1}\right\},$$

in which the inequality and the last equation are suggested by (S.1) and (S.7) respectively. □

S.3 Proof of Theorem 2

Proof. Denote matrix $(\boldsymbol{\phi}_k)_{N\times L} = \{\boldsymbol{\phi}_k(1/N), \ldots, \boldsymbol{\phi}_k(N/N)\}^\top$ for $k \geq 1$, and denote the vector-valued function

$$\tilde{\boldsymbol{\phi}}_k(\cdot)^\top = N^{-1}\boldsymbol{B}(\cdot)^\top \boldsymbol{V}_p^{-1}\boldsymbol{X}^\top (\boldsymbol{\phi}_k)_{N\times L}.$$

According to (S.2), $\tilde{\boldsymbol{\eta}}_{it}(\cdot) = \tilde{\boldsymbol{m}}(\cdot) + \sum_{k=1}^{\infty} \boldsymbol{\xi}_{itk} \circ \tilde{\boldsymbol{\phi}}_k(\cdot)$, therefore,

$$\tilde{\boldsymbol{\eta}}_{it}(\cdot) - \boldsymbol{\eta}_{it}(\cdot) = \tilde{\boldsymbol{m}}(\cdot) - \boldsymbol{m}(\cdot) + \sum_{k=1}^{\infty} \boldsymbol{\xi}_{itk} \circ \left\{\tilde{\boldsymbol{\phi}}_k(\cdot) - \boldsymbol{\phi}_k(\cdot)\right\}.$$

Lemma A.4 of Cao et al. (2012) provides a constant $C_{q,\mu} > 0$, such that

$$\sup_{x\in[0,1]} \|\tilde{\boldsymbol{m}}(x) - \boldsymbol{m}(x)\|_\infty \leq C_{q,\mu} \|\boldsymbol{m}\|_{q,\mu} J_s^{-p^*}, \tag{S.8}$$

$$\sup_{x\in[0,1]} \left\|\tilde{\boldsymbol{\phi}}_k(x) - \boldsymbol{\phi}_k(x)\right\|_\infty \leq C_{q,\mu} \|\boldsymbol{\phi}_k\|_{q,\mu} J_s^{-p^*}, \quad k \geq 1, \tag{S.9}$$

which lead to

$$\begin{aligned} &\sup_{x\in[0,1]} \|\tilde{\boldsymbol{\eta}}_{it}(x) - \boldsymbol{\eta}_{it}(x)\|_\infty \\ &\leq \sup_{x\in[0,1]} \|\tilde{\boldsymbol{m}}(x) - \boldsymbol{m}(x)\|_\infty + \sum_{k=1}^{\infty} \|\boldsymbol{\xi}_{itk}\|_\infty \times \sup_{x\in[0,1]} \left\|\tilde{\boldsymbol{\phi}}_k(x) - \boldsymbol{\phi}_k(x)\right\|_\infty \\ &\leq C_{q,\mu} J_s^{-p^*} W_{it}, \end{aligned}$$

where $W_{it,\xi} = \|\boldsymbol{m}\|_{q,\mu} + \sum_{k=1}^{\infty} \|\boldsymbol{\xi}_{itk}\|_{\infty} \times \|\boldsymbol{\phi}_k\|_{q,\mu}$. Note that Assumption (A4) entails that $\mathbb{E}W_{it,\xi}^{r_0} < \infty$, then

$$\mathbb{P}\left\{\max_{1\leq r(i,t)\leq nT} W_{it,\xi} > (nT\log nT)^{2/r_0}\right\} \leq \mathbb{E}W_{it,\xi}^{r_0} (nT)^{-1} (\log nT)^{-2}.$$

Since $\sum_{s=1}^{\infty} s^{-1} (\log s)^{-2} < +\infty$, then $\max_{1\leq r(i,t)\leq nT} W_{it,\xi} = \mathcal{O}_{a.s.}\left\{(nT\log nT)^{2/r_0}\right\}$ by Borel-Cantelli Lemma, hence

$$\sup_{x\in[0,1]} \left\|(nT)^{-1} \sum_{r(i,t)=1}^{nT} \{\tilde{\boldsymbol{\eta}}_{it}(x) - \boldsymbol{\eta}_{it}(x)\}\right\|_{\infty} \leq \max_{1\leq r(i,t)\leq nT} \sup_{x\in[0,1]} \|\tilde{\boldsymbol{\eta}}_{it}(x) - \boldsymbol{\eta}_{it}(x)\|_{\infty}$$
$$= \mathcal{O}_{a.s.}\{J_s^{-p^*} (nT\log nT)^{2/r_0}\}.$$

Together with Lemma S.8 and decomposition in (S.3), one obtains that

$$\sup_{x\in[0,1]} \|\bar{\boldsymbol{m}}(x) - \hat{\boldsymbol{m}}(x)\|_{\infty}$$
$$= \mathcal{O}_{a.s.}\left\{J_s^{-p^*} (nT\log nT)^{2/r_0} + (nT)^{-1/2} N^{-1/2} J_s^{1/2} \log^{1/2} N + N^{\beta_2 - 1} J_s\right\},$$

then orders of nT and J_s relative to N in Assumptions (A3) and (A6) imply that

$$\sup_{x\in[0,1]} (nT)^{1/2} \|\bar{\boldsymbol{m}}(x) - \hat{\boldsymbol{m}}(x)\|_{\infty} = o_p(1).$$

□

S.4 Proof of Theorem 1

Proof. Define an ℓ^2-random variable by

$$\bar{Z}_{nT,\xi} = \left\{ (nT)^{1/2} \left(\boldsymbol{\lambda}_1^{1/2} \circ \bar{\boldsymbol{Z}}_{\cdot\cdot 1,\xi} \right)^\top, (nT)^{1/2} \left(\boldsymbol{\lambda}_2^{1/2} \circ \bar{\boldsymbol{Z}}_{\cdot\cdot 2,\xi} \right)^\top, \ldots \right\}^\top,$$

with $\bar{\boldsymbol{Z}}_{\cdot\cdot k,\xi} = (nT)^{-1} \sum_{r(i,t)=1}^{nT} \bar{\boldsymbol{Z}}_{itk,\xi}$. Recalling the definition of $\Pi : \ell^2 \to (\mathcal{L}^2[0,1])^L$ in (2.18), one has

$$\Pi(\bar{Z}_{nT,\xi})(\cdot) = \sum_{k=1}^{\infty} (nT)^{1/2} \boldsymbol{\lambda}_k^{1/2} \circ \bar{\boldsymbol{Z}}_{\cdot\cdot k,\xi} \circ \boldsymbol{\psi}_k(\cdot) = \sum_{k=1}^{\infty} (nT)^{1/2} \bar{\boldsymbol{Z}}_{\cdot\cdot k,\xi} \circ \boldsymbol{\phi}_k(\cdot).$$

Denote $\boldsymbol{\varphi}_{nT}(\cdot) = \boldsymbol{G}_\varphi^{-1/2}(\cdot,\cdot)\Pi(\bar{Z}_{nT,\xi})(\cdot)$, then for any positive integer d, $(x_1, \ldots, x_d) \in [0,1]^d$, $(c_1, \ldots, c_d)^\top \in \mathbb{R}^d$, and $\boldsymbol{b} = (b_1, \ldots, b_L)^\top \in \mathbb{R}^L$ with $\|\boldsymbol{b}\|_2 = 1$,

$$\begin{aligned} &\lim_{T\to\infty} \mathrm{Var} \left\{ \sum_{s=1}^{d} c_s \boldsymbol{b}^\top \boldsymbol{\varphi}_{nT}(x_s) \right\} \\ &= \lim_{T\to\infty} \sum_{s,s'=1}^{d} \mathrm{Cov} \left\{ c_s \boldsymbol{b}^\top \boldsymbol{\varphi}_{nT}(x_s), c_{s'} \boldsymbol{b}^\top \boldsymbol{\varphi}_{nT}(x_{s'}) \right\} \\ &= \sum_{s,s'=1}^{d} c_s c_{s'} \boldsymbol{b}^\top \left[\lim_{T\to\infty} \mathbb{E} \left\{ \boldsymbol{\varphi}_{nT}(x_s) \boldsymbol{\varphi}_{nT}(x_{s'})^\top \right\} \right] \boldsymbol{b}. \end{aligned}$$

Noting that $\lim_{T\to\infty} nT\, \mathbb{E} \bar{\boldsymbol{Z}}_{\cdot\cdot k,\xi} \bar{\boldsymbol{Z}}_{\cdot\cdot k,\xi}^\top = \sum_{h=-\infty}^{\infty} \mathbb{E} \boldsymbol{\xi}_{itk} \boldsymbol{\xi}_{i,t+h,k}^\top = \boldsymbol{\Delta}_k$, one has

$$\lim_{T\to\infty} \mathbb{E} \left\{ \boldsymbol{\varphi}_{nT}(x_s) \boldsymbol{\varphi}_{nT}(x_{s'})^\top \right\} = \boldsymbol{G}_\varphi^{-1/2}(x_s, x_s)\, \boldsymbol{G}_\varphi(x_s, x_{s'})\, \boldsymbol{G}_\varphi^{-1/2}(x_{s'}, x_{s'}),$$

thus

$$\begin{aligned}
&\lim_{T\to\infty} \mathrm{Var}\left\{\sum_{s=1}^{d} c_s \boldsymbol{b}^\top \boldsymbol{\varphi}_{nT}(x_s)\right\}\\
&= \sum_{s,s'=1}^{d} c_s c_{s'} \boldsymbol{b}^\top \ \boldsymbol{G}_\varphi^{-1/2}(x_s,x_s)\boldsymbol{G}_\varphi(x_s,x_{s'})\boldsymbol{G}_\varphi^{-1/2}(x_{s'},x_{s'})\boldsymbol{b}\\
&= \ \mathrm{Var}\left\{\sum_{s=1}^{d} c_s \boldsymbol{b}^\top \boldsymbol{\varphi}\left(x_s\right)\right\},
\end{aligned}$$

with the Gaussian process $\boldsymbol{\varphi}(\cdot)$ defined in (2.19). Hence

$$\left\{\boldsymbol{b}^\top\boldsymbol{\varphi}_{nT}(x_1),\ldots,\boldsymbol{b}^\top\boldsymbol{\varphi}_{nT}(x_d)\right\} \to_D \left\{\boldsymbol{b}^\top\boldsymbol{\varphi}(x_1),\ldots,\boldsymbol{b}^\top\boldsymbol{\varphi}(x_d)\right\}, \quad T\to\infty. \tag{S.10}$$

Under Assumption (A4), there exists $c_\varphi > 0$ such that $\boldsymbol{G}_\varphi\left(x,x\right) \geq c_\varphi \boldsymbol{I}$, $x\in[0,1]$, then by the definition of $\omega(\boldsymbol{\varphi}_{nT},\delta)$,

$$\begin{aligned}
\omega(\boldsymbol{b}^\top\boldsymbol{\varphi}_{nT},\delta) &= \sup_{x,x'\in[0,1],|x-x'|\leq\delta} \left|\boldsymbol{b}^\top\boldsymbol{\varphi}_{nT}\left(x\right) - \boldsymbol{b}^\top\boldsymbol{\varphi}_{nT}(x')\right|\\
&\leq \sup_{x,x'\in[0,1],|x-x'|\leq\delta} \|\boldsymbol{b}\|_2 \left\|\boldsymbol{G}_\varphi^{-1/2}(x,x)\left\{\Pi\left(\bar{Z}_{nT,\xi}\right)(x) - \Pi\left(\bar{Z}_{nT,\xi}\right)(x')\right\}\right\|_2\\
&\leq \sup_{x,x'\in[0,1],|x-x'|\leq\delta} c_\varphi^{-1/2} \left\|\Pi\left(\bar{Z}_{nT,\xi}\right)(x) - \Pi\left(\bar{Z}_{nT,\xi}\right)(x')\right\|_2,
\end{aligned}$$

thus

$$\begin{aligned}
&\mathbb{P}\left\{\omega(\boldsymbol{b}^{\top}\boldsymbol{\varphi}_{nT},\delta)\geq\epsilon\right\}\\
&\leq\mathbb{P}\left\{\sup_{x,x'\in[0,1],|x-x'|\leq\delta}c_{\varphi}^{-1/2}\left\|\Pi\left(\bar{Z}_{nT,\xi}\right)(x)-\Pi\left(\bar{Z}_{nT,\xi}\right)(x')\right\|_{2}\geq\epsilon\right\}\\
&\leq\epsilon^{-1}c_{\varphi}^{-1/2}\mathbb{E}\sup_{x,x'\in[0,1],|x-x'|\leq\delta}\left\|\sum_{k=1}^{\infty}(nT)^{1/2}\bar{\boldsymbol{Z}}_{\cdot\cdot k,\xi}\circ\left\{\boldsymbol{\phi}_{k}(x)-\boldsymbol{\phi}_{k}\left(x'\right)\right\}\right\|_{2}\\
&\leq\epsilon^{-1}c_{\varphi}^{-1/2}\delta^{\mu}\sum_{k=1}^{\infty}\|\boldsymbol{\phi}_{k}\|_{0,\mu}\left\{(nT)^{1/2}\mathbb{E}\left\|\bar{\boldsymbol{Z}}_{\cdot\cdot k,\xi}\right\|_{2}\right\}.
\end{aligned}$$

Noting that

$$\begin{aligned}
\lim_{T\to\infty}(nT)\mathbb{E}\left\|\bar{\boldsymbol{Z}}_{\cdot\cdot k,\xi}\right\|_{2}^{2}&=\lim_{T\to\infty}\left\|\sum_{t=0}^{\infty}\boldsymbol{a}_{tk}\circ\boldsymbol{a}_{tk}+2\sum_{h=1}^{T-1}\sum_{t=0}^{\infty}\frac{T-h}{T}\boldsymbol{a}_{tk}\circ\boldsymbol{a}_{t+h,k}\right\|_{2}^{2}\\
&=\left\|\left(\sum_{t=0}^{\infty}\boldsymbol{a}_{tk}\right)\circ\left(\sum_{t=0}^{\infty}\boldsymbol{a}_{tk}\right)\right\|_{2}^{2}
\end{aligned}$$

is uniformly bounded for $k\geq 1$, with Assumption $\sum_{k=1}^{\infty}\|\boldsymbol{\phi}_{k}\|_{0,\mu}<\infty$, one has

$$\lim_{\delta\to 0}\limsup_{T\to\infty}\epsilon^{-1}c_{\varphi}^{-1/2}\delta^{\mu}\sum_{k=1}^{\infty}\|\boldsymbol{\phi}_{k}\|_{0,\mu}(nT)^{1/2}\,\mathbb{E}\left\|\bar{\boldsymbol{Z}}_{\cdot\cdot k,\xi}\right\|_{2}=0,$$

and

$$\lim_{\delta\to 0}\limsup_{T\to\infty}\mathbb{P}\left\{\omega(\boldsymbol{b}^{\top}\boldsymbol{\varphi}_{nT},\delta)\geq\varepsilon\right\}=0.$$

According to (S.10) and Lemma S.2, one has that $\boldsymbol{b}^{\top}\boldsymbol{\varphi}_{nT}\to_{D}\boldsymbol{b}^{\top}\boldsymbol{\varphi}$ holds for any $\boldsymbol{b}\in\mathbb{R}^{L}$ with $\boldsymbol{b}^{\top}\boldsymbol{b}=1$. Since $\boldsymbol{\varphi}_{nT}\in(\mathcal{D}[0,1])^{L}$ and $\boldsymbol{\varphi}\in(\mathcal{C}[0,1])^{L}$

$a.s.$, according to Lemma S.3, $\boldsymbol{\varphi}_{nT} \to_D \boldsymbol{\varphi}$.

Define

$$\boldsymbol{\varpi}_{1,nT}(\cdot) = \sum_{k=1}^{k_{nT}} \left(\bar{\boldsymbol{\xi}}_{\cdot\cdot k} - \bar{\boldsymbol{Z}}_{\cdot\cdot k,\xi}\right) \circ \boldsymbol{\phi}_k(\cdot), \ \boldsymbol{\varpi}_{2,nT}(\cdot) = \sum_{k=k_{nT}+1}^{\infty} \left(\bar{\boldsymbol{\xi}}_{\cdot\cdot k} - \bar{\boldsymbol{Z}}_{\cdot\cdot k,\xi}\right) \circ \boldsymbol{\phi}_k(\cdot),$$

where $\bar{\boldsymbol{\xi}}_{\cdot\cdot k} = (nT)^{-1} \sum_{r(i,t)=1}^{nT} \boldsymbol{\xi}_{itk}$. Lemma S.6 suggests that

$$\mathbb{P}\left\{ \max_{1 \le k \le k_{nT}} \left\| \bar{\boldsymbol{\xi}}_{\cdot\cdot k} - \bar{\boldsymbol{Z}}_{\cdot\cdot k,\xi} \right\|_2 > C_3 (nT)^{\beta_1 - 1} \right\} < C_4 (nT)^{-\gamma_3},$$

then by Borel-Cantelli Lemma, one has

$$\max_{1 \le k \le k_{nT}} \left\| \bar{\boldsymbol{\xi}}_{\cdot\cdot k} - \bar{\boldsymbol{Z}}_{\cdot\cdot k,\xi} \right\|_2 = \mathcal{O}_{a.s.}\left\{ (nT)^{\beta_3 - 1} \right\},$$

then

$$\begin{aligned}
\sup_{x \in [0,1]} \left\| (nT)^{1/2} \boldsymbol{G}_{\varphi}^{-1/2}(x,x) \boldsymbol{\varpi}_{1,nT}(x) \right\|_\infty &\le (nT)^{1/2} \sup_{x \in [0,1]} \left\| \boldsymbol{G}_{\varphi}^{-1/2}(x,x) \boldsymbol{\varpi}_{1,nT}(x) \right\|_2 \\
&\le c_{\varphi}^{-1/2} (nT)^{1/2} \sup_{x \in [0,1]} \|\boldsymbol{\varpi}_{1,nT}(x)\|_2 \\
&\le c_{\varphi}^{-1/2} (nT)^{1/2} \max_{1 \le k \le k_{nT}} \left\| \bar{\boldsymbol{\xi}}_{\cdot\cdot k} - \bar{\boldsymbol{Z}}_{\cdot\cdot k,\xi} \right\|_2 \sum_{k=1}^{k_{nT}} \|\boldsymbol{\phi}_k\|_\infty \\
&= \mathcal{O}_{a.s}\left\{ (nT)^{\beta_3 - 1/2} \right\} = o_{a.s.}(1).
\end{aligned}$$

Similarly,

$$\sup_{x\in[0,1]}\left\|(nT)^{1/2}\boldsymbol{G}_{\varphi}^{-1/2}(x,x)\boldsymbol{\varpi}_{2,nT}(x)\right\|_{\infty}\le c_{\varphi}^{-1/2}(nT)^{1/2}\sup_{x\in[0,1]}\|\boldsymbol{\varpi}_{2,nT}(x)\|_2$$
$$\le c_{\varphi}^{-1/2}(nT)^{1/2}\sum_{k=k_{nT}+1}^{\infty}\|\boldsymbol{\phi}_k\|_{\infty}\left\|\bar{\boldsymbol{\xi}}_{\cdot\cdot k}-\bar{\boldsymbol{Z}}_{\cdot\cdot k,\xi}\right\|_2.$$

Since $(nT)^{1/2}\mathbb{E}\left\|\bar{\boldsymbol{\xi}}_{\cdot\cdot k}-\bar{\boldsymbol{Z}}_{\cdot\cdot k,\xi}\right\|_2\le 2(nT)^{1/2}\mathbb{E}\left\|\bar{\boldsymbol{\xi}}_{\cdot\cdot k}\right\|_2$ are uniformly bounded for $k>k_{nT}$ and $\sum_{k=k_{nT}+1}^{\infty}\|\boldsymbol{\phi}_k\|_{\infty}=o(1)$ by Assumption (A4),

$$\sup_{x\in[0,1]}\left\|(nT)^{1/2}\boldsymbol{G}_{\varphi}^{-1/2}(x,x)\boldsymbol{\varpi}_{2,nT}(x)\right\|_{\infty}=\mathcal{O}_p(1),$$

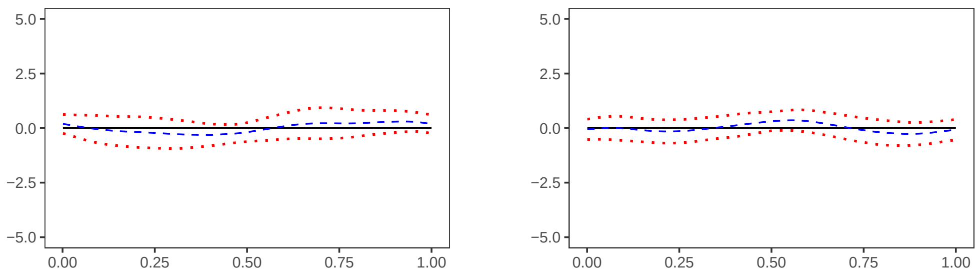
hence

$$\left\|\sup_{x\in[0,1]}(nT)^{1/2}\boldsymbol{G}_{\varphi}^{-1/2}(x,x)\left\{\boldsymbol{\varpi}_{1,nT}(x)+\boldsymbol{\varpi}_{2,nT}(x)\right\}\right\|_{\infty}=\mathcal{O}_p(1).$$

Finally by Slutsky's theorem and the equation that

$$(nT)^{1/2}\boldsymbol{G}_{\varphi}^{-1/2}(\cdot,\cdot)\left\{\bar{\boldsymbol{m}}(\cdot)-\boldsymbol{m}(\cdot)\right\}=\boldsymbol{\varphi}_{nT}(\cdot)+(nT)^{1/2}\boldsymbol{G}_{\varphi}^{-1/2}(\cdot,\cdot)\left\{\boldsymbol{\varpi}_{1,nT}(\cdot)+\boldsymbol{\varpi}_{2,nT}(\cdot)\right\},$$

one completes the proof. □



(a) (b)

Figure S.1: Plots of the true $m^{(1)}(\cdot) - m^{(2)}(\cdot) \equiv 0$ (solid) and cubic spline estimator $\hat{m}^{(1)}(\cdot) - \hat{m}^{(2)}(\cdot)$ (dashed), with 95% SCB (dotted). The number N of observations in (a) and (b) are 900 and 1600 respectively.

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Table S.1: Coverage frequencies for $m^{(1)}(\cdot) \equiv (1,0,0)\boldsymbol{m}(\cdot)$ by SCB (3.27).

$\boldsymbol{\zeta}_{itk}$	$\boldsymbol{\Sigma}_{\zeta,k}$	N	$1-\alpha$	$\varepsilon_{itj}^{(l)} \sim N(0,1)$		$\varepsilon_{itj}^{(l)} \sim \mathrm{U}(-\sqrt{3},\sqrt{3})$	
				$\boldsymbol{\sigma}_{\text{homo}}$	$\boldsymbol{\sigma}_{\text{hetero}}$	$\boldsymbol{\sigma}_{\text{homo}}$	$\boldsymbol{\sigma}_{\text{hetero}}$
Normal	AR-1	900	0.95	0.938	0.942	0.939	0.938
			0.99	0.984	0.985	0.985	0.989
		1600	0.95	0.926	0.928	0.931	0.924
			0.99	0.986	0.985	0.983	0.988
	TOEP	900	0.95	0.941	0.937	0.943	0.941
			0.99	0.987	0.989	0.989	0.987
		1600	0.95	0.957	0.953	0.956	0.958
			0.99	0.994	0.994	0.996	0.992
Student's t	AR-1	900	0.95	0.932	0.931	0.928	0.931
			0.99	0.98	0.977	0.981	0.981
		1600	0.95	0.939	0.938	0.938	0.931
			0.99	0.984	0.982	0.986	0.983
	TOEP	900	0.95	0.939	0.94	0.937	0.937
			0.99	0.985	0.988	0.986	0.986
		1600	0.95	0.931	0.93	0.929	0.935
			0.99	0.984	0.982	0.984	0.985

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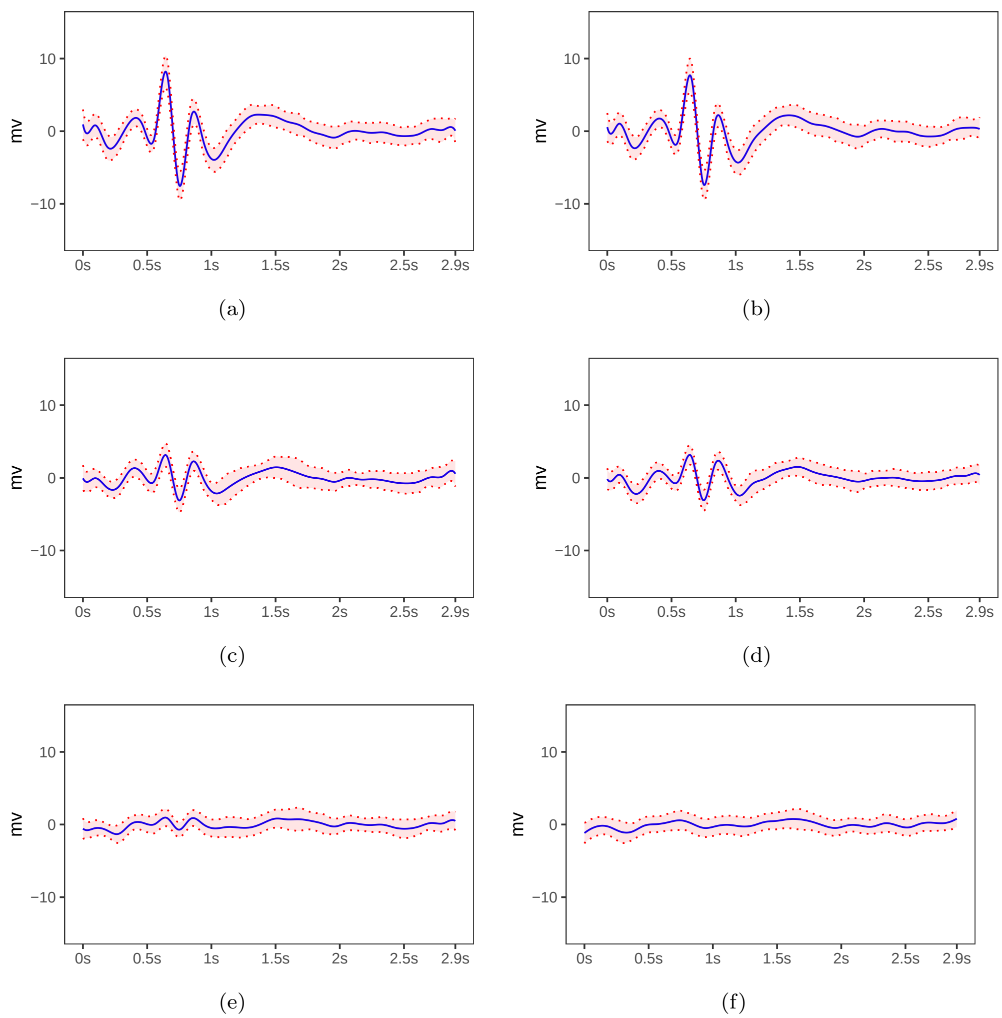

Figure S.2: Plots of cubic spline estimator (solid) with 95% SCB (dotted) for $m^{(l)}(\cdot), l = 1, \ldots, 6$ of group 1, in which (a)-(f) correspond to $m^{(O_1)}(\cdot), m^{(O_2)}(\cdot), m^{(P_3)}(\cdot), m^{(P_4)}(\cdot), m^{(C_3)}(\cdot), m^{(F_3)}(\cdot)$ respectively.

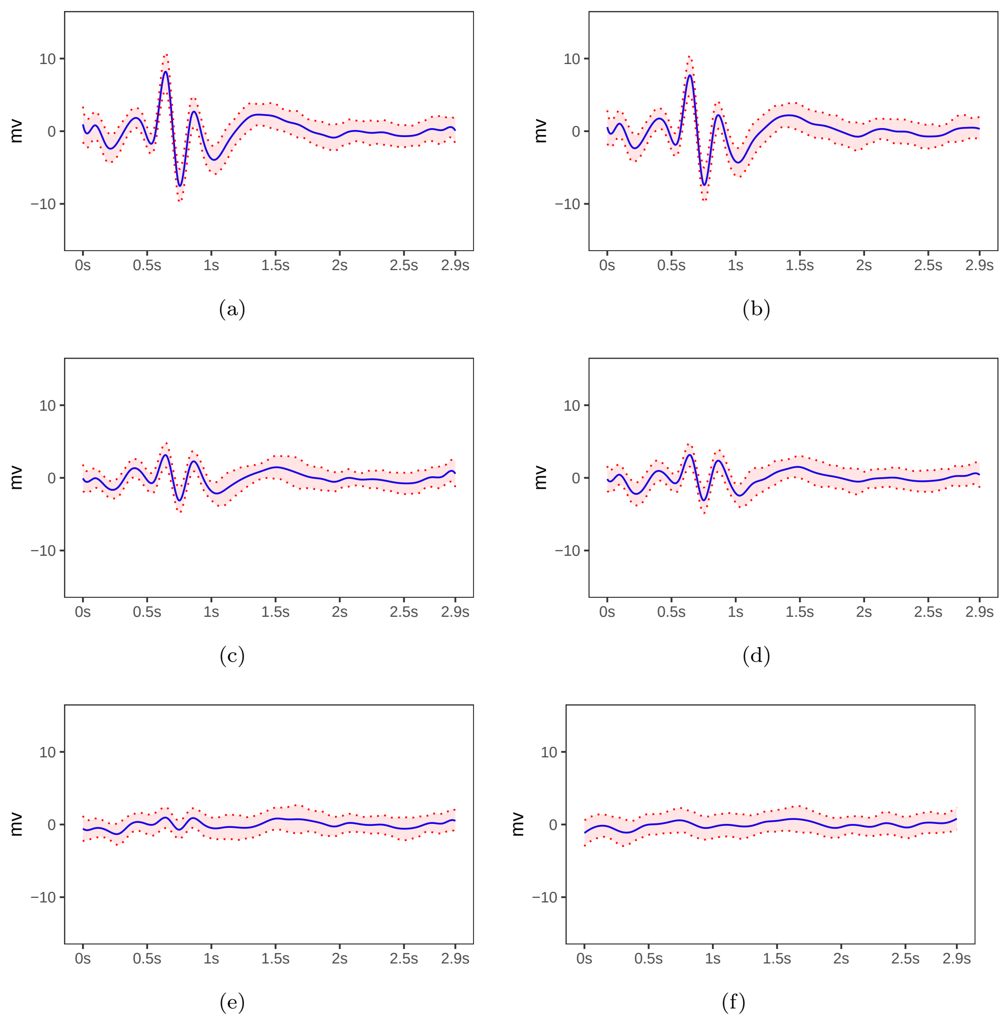

Figure S.3: Plots of cubic spline estimator (solid) with 95% SCR (dotted) for $\boldsymbol{m}(\cdot)$ of group 1, in which (a)-(f) correspond to $m^{(O_1)}(\cdot)$, $m^{(O_2)}(\cdot)$, $m^{(P_3)}(\cdot)$, $m^{(P_4)}(\cdot)$, $m^{(C_3)}(\cdot)$ and $m^{(F_3)}(\cdot)$ respectively.