HYPOTHESES TESTING OF FUNCTIONAL PRINCIPAL COMPONENTS

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Supplementary Material

This supplement provides technical lemmas and detailed proofs of the main asymptotic results. Throughout this section, $\mathcal{O}_p$ (or o_p) denotes a sequence of random variables of certain order in probability. For instance, $o_p(n^{-1/2})$ means a smaller order than $n^{-1/2}$ in probability, and by $\mathcal{O}_{a.s.}$ (or $o_{a.s.}$) almost surely $\mathcal{O}$ (or o).

For any $\mathcal{L}^2$ integrable functions $\phi(x)$ and $\varphi(x)$, $x \in [0,1]$, denote by $\langle \phi, \varphi \rangle = \int_{[0,1]} \phi(x) \varphi(x) \, dx$ their theoretical inner product, and $\langle \phi, \varphi \rangle_N = N^{-1} \sum_{j=1}^{N} \phi(j/N) \varphi(j/N)$ the empirical inner product. The related theoretical and empirical norms are $\|\phi\|_2^2 = \langle \phi, \phi \rangle$, $\|\phi\|_{2,N}^2 = \langle \phi, \phi \rangle_N$. For any vector $\mathbf{a} = (a_1, \ldots, a_n)^{\intercal} \in \mathbb{R}^n$, denote the norm $\|\mathbf{a}\|_r = (|a_1|^r + \cdots + |a_n|^r)^{1/r}$, $1 \leq r < +\infty$, $\|\mathbf{a}\|_\infty = \max(|a_1|, \ldots, |a_n|)$. For any matrix $\mathbf{A} = (a_{ij})_{i=1,j=1}^{m,n}$, denote its L_r norm as $\|\mathbf{A}\|_r = \max_{\mathbf{a} \in \mathbb{R}^n, \mathbf{a} \neq \mathbf{0}} \|\mathbf{A}\mathbf{a}\|_r \, \|\mathbf{a}\|_r^{-1}$, for $r < +\infty$ and $\|\mathbf{A}\|_r = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}|$, for $r = \infty$. For any random variable $\mathbf{X}$, if it is L_p-integrable, denotes its L_p norm as $\|\mathbf{X}\|_p = (\mathbb{E}|\mathbf{X}|^p)^{1/p}$.

S.1 Technical Lemmas

Lemma S.1. *[Theorem 2.6.7 of Csőrgő and Révész (1981)] Suppose that* $\{\xi_i, i \in \mathbb{N}_+\}$ *are i.i.d. on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with* $\mathbb{E}(\xi_1) = 0$, $\mathbb{E}(\xi_1^2) = 1$, *and* $H(x) > 0$ $(x \geq 0)$ *is an increasing continuous function such that* $x^{-2-\gamma} H(x)$ *is increasing for some* $\gamma > 0$ *and* $x^{-1} \log H(x)$ *is decreasing with* $\mathbb{E} H(|\xi_1|) < \infty$. *Then there exist constants* C_1, C_2, $a > 0$ *which depend only on the distribution of* ξ_1, *a new probability space* $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}\right)$ *on which are* $\left\{\tilde{\xi}_i, i \in \mathbb{N}_+\right\} \stackrel{D}{=} \{\xi_i, i \in \mathbb{N}_+\}$ *and a Brownian motion* $W(t)$, *such that for* $\{x_n\}_{n=1}^{\infty}$ *satisfying* $H^{-1}(n) < x_n < C_1 (n \log n)^{1/2}$, $\tilde{\mathbb{P}}\left\{\max_{1 \leq m \leq n} \left|\tilde{S}_m - W(m)\right| > x_n\right\} \leq C_2 n \{H(a x_n)\}^{-1}$ *where* $\tilde{S}_m = \sum_{i=1}^m \tilde{\xi}_i$.

For simplicity, we abuse notation à la Csőrgő and Révész (1981), by abandoning the precise statement in Lemma S.1 about partial sum $\tilde{S}_m$ on new probability space $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}\right)$, and stating instead the following for $S_m = \sum_{i=1}^m \xi_i$ on the original $(\Omega, \mathcal{F}, \mathbb{P})$

$$\mathbb{P}\left\{\max_{1 \leq m \leq n} |S_m - W(m)| > x_n\right\} \leq C_2 n \{H(a x_n)\}^{-1}.$$

Lemma S.2. *The distribution function of* S *is continuous and strictly increasing.*

Proof. As $\sum_{1 \leq k < k' < \infty} \lambda_k \lambda_{k'} < \infty$, Zolotarev (1961) and Hoeffding (1964) ensure that S has a Lebesgue density function over $(0, +\infty)$, so its distribution function is continuous. In addition, equations (2) and (4) of Christoph et al. (1996) provide that the density function of S is positive on $(0,+\infty)$, sandwiched by constant multiples of the density function of any finite partial sum. Thus the distribution function of S is strictly increasing. The proof is completed. □

Lemma S.3. *[Lemma A.12 of Wang et al. (2020)] Let* $W_i \sim N\left(0,\sigma_i^2\right), \sigma_i > 0$, $i = 1,\ldots,n$, *then for* $n > 2, a > 2$,

$$\mathbb{P}\left(\max_{1\le i\le n}|W_i/\sigma_i| > a\sqrt{\log n}\right) < 2n^{1-a^2/2}.$$

Hence, $\left(\max_{1\le i\le n}|W_i|\right)/\left(\max_{1\le i\le n}\sigma_i\right) \le \max_{1\le i\le n}|W_i/\sigma_i| = \mathcal{O}_{a.s.}\left(\sqrt{\log n}\right)$. *Consequently, for* n *random variables* $\chi_i \sim \chi^2(1)$, $\max_{1\le i\le n}\chi_i = \mathcal{O}_{a.s.}(\log n)$.

Lemma S.4. *[Theorem 1.2 of Bosq (1998)] Let* $X_1,\ldots,X_n$ *be independent zero-mean real-valued random variables and let* $S_n = \sum_{i=1}^n X_i$. *If there exists* $c > 0$ *such that*

$$\mathbb{E}|X_i|^k \le c^{k-2}k!\mathbb{E}X_i^2 < +\infty, i = 1,\ldots,n, k = 3,4,\ldots$$

(Cramér's conditions) then

$$\mathbb{P}\left(|S_n| \ge t\right) \le 2\exp\left(-\frac{t^2}{4\sum_{i=1}^n \mathbb{E}X_i^2 + 2ct}\right), t > 0$$

(Bernstein's inequality).

Lemma S.5. *For κ_n in (2.34), let $R_{kk',n}, 1 \leq k < k', k' > \kappa_n$ be an array of nonnegative-valued random variables with $\max_{1\leq k<k',k'>\kappa_n} \mathbb{E}R_{kk',n} \leq M$, then* $\sum_{1\leq k<k',k'>\kappa_n} \lambda_k\lambda_{k'} R_{kk',n} = o_p(1)$.

Proof. One simply takes expectation of the non-negative random variable $\sum_{1\leq k<k',k'>\kappa_n} \lambda_k\lambda_{k'} R_{kk',n}$

$$\begin{aligned}\mathbb{E}\sum_{1\leq k<k',k'>\kappa_n} \lambda_k\lambda_{k'} R_{kk',n} &= \sum_{1\leq k<k',k'>\kappa_n} \lambda_k\lambda_{k'}\mathbb{E}R_{kk',n} \leq \\ M\sum_{1\leq k<k',k'>\kappa_n} \lambda_k\lambda_{k'} &\leq M\sum_{k=1}^{\infty}\lambda_k \sum_{k'=\kappa_n+1}^{\infty}\lambda_{k'} = o(1).\end{aligned}$$

By (2.34), $\kappa_n \to \infty$, so $\sum_{k'=\kappa_n+1}^{\infty} \lambda_{k'} \to 0$, and hence $\sum_{1\leq k<k',k'>\kappa_n} \lambda_k\lambda_{k'} R_{kk',n} = o_p(1)$. □

Lemma S.6. *Under Assumptions (A1), (B1), there exists a $\mathcal{C}[0,1]$-valued mean zero Gaussian random variable $\mathcal{N}_\xi$ such that as $n \to \infty$,*

$$n^{1/2}\left(n^{-1}\sum\nolimits_{i=1}^{n} \xi_i\right) \xrightarrow{D} \mathcal{N}_\xi, \left\|n^{-1}\sum\nolimits_{i=1}^{n} \xi_i\right\|_\infty = \mathcal{O}_p\left(n^{-1/2}\right).$$

Under Assumptions (A1), (B1)-(B2), as $n \to \infty$

$$\max_{1\leq i\leq n}\left\|\hat{\xi}_i - \xi_i\right\|_\infty = \mathcal{O}_p\left(n^{-1/2}\right). \tag{S.1}$$

Proof. Assumptions (A1) and (B1) ensure that ξ_1 is a mean zero $\mathcal{C}[0,1]$ random variable, whose Lipschitz seminorm has finite second moment, so according to Theorem 2.8 of Bosq (2000), $n^{1/2}(n^{-1}\sum_{i=1}^{n}\xi_i)$ as sample mean of i.i.d. $\mathcal{C}[0,1]$-valued mean zero random variables converge in distribution to a $\mathcal{C}[0,1]$-valued mean zero Gaussian random variable $\mathcal{N}_\xi$. Continuous Mapping Theorem then implies that $\left\|n^{1/2}\left(n^{-1}\sum_{i=1}^n \xi_i\right)\right\|_\infty \xrightarrow{D} \|\mathcal{N}_\xi\|_\infty$, hence $\left\|n^{1/2}\left(n^{-1}\sum_{i=1}^n \xi_i\right)\right\|_\infty = \mathcal{O}_p(1)$. This, together with Assumption (B2) lead to (S.39). □

Lemma S.7. *(i) If $\{\kappa_n\}_{n=1}^\infty$ satisfy (2.34) and positive sequences $\left\{\rho_{n,N}\right\}_{n=1}^\infty$ $\kappa_n^2 n^{1/2}\rho_{n,N}\log^{1/2} n \to 0$, then*

$$\kappa_n = o\left(n^{1/4}\right), \rho_{n,N} = o\left(n^{-1/2}\right), \tag{S.2}$$

$$\kappa_n^2 n\rho_{n,N}^2 \to 0. \tag{S.3}$$

(ii) For positive integers $\{\kappa_n\}_{n=1}^\infty$ with $\kappa_n^2 = o\left(n^{1/2}\right)$ and $r_1 > 6$, there exist constants $\alpha_1 \in (3/2r_1, 1/2), \alpha_2 \in (3/r_1, 1/2)$, such that for $D_n = n^{\alpha_1}, D_n' = n^{\alpha_2}$, the following are fulfilled:

$$D_n^{-(r_1-1)} n^{1/2}\log^{-1/2} n \to 0, \tag{S.4}$$

$$\sum_{n=1}^\infty \kappa_n^2 D_n^{-r_1} < \infty, \tag{S.5}$$

$$D_n\sqrt{\log n}\, n^{-1/2} \to 0, \tag{S.6}$$

$$D_n'^{-(r_1/2-1)} n^{1/2}\log^{-1/2} n \to 0, \tag{S.7}$$

$$\sum_{n=1}^{\infty} \kappa_n^2 D_n'^{-r_1/2} < \infty, \tag{S.8}$$

$$D_n' \sqrt{\log n} n^{-1/2} \to 0. \tag{S.9}$$

PROOF. (i) $\kappa_n^2 n^{-1/2} \log^{3/2} n \to 0$ leads to $\kappa_n^2 n^{-1/2} \to 0$, hence $\kappa_n = o\left(n^{1/4}\right)$. Likewise, $\rho_{n,N} = o\left(n^{-1/2}\right)$ since $\kappa_n \to \infty$ and $\kappa_n^2 \rho_{n,N} n^{1/2} \log^{1/2} n \to 0$, thus (S.2) holds. Under the constraints of κ_n and $\rho_{n,N}$,

$$\kappa_n^2 n \rho_{n,N}^2 = \left(\kappa_n^2 \rho_{n,N} n^{1/2} \log^{1/2} n\right)^2 \times \kappa_n^{-2} \times \log^{-1} n \to 0,$$

thus (S.3) holds.

(ii) Since $\alpha_1 > 3/2r_1$ and $r_1 > 3/2$ by (2.44), $\alpha_1 > 1/\left(2r_1 - 2\right)$, then $-\alpha_1\left(r_1 - 1\right) + 1/2 < 0$. Thus,

$$D_n^{-(r_1-1)} n^{1/2} \log^{-1/2} n = n^{-\alpha_1(r_1-1)+1/2} \log^{-1/2} n \to 0,$$

(S.4) holds. Similarly, since $\alpha_2 > 3/r_1$ and $r_1 > 3$ by (2.44), (S.7) can be proved.

Since $\alpha_1 > 3/2r_1$, $D_n^{-r_1} = n^{-\alpha_1 r_1} = o\left(n^{-3/2}\right)$. Thus $\kappa_n^2 = o\left(n^{1/2}\right)$ implies $\kappa_n^2 D_n^{-r_1} = o\left(n^{-1}\right)$, so $\sum_{n=1}^{\infty} \kappa_n^2 D_n^{-r_1} < \infty$, which yields (S.5). (S.8) can be proved in a similar way.

Note that $\alpha_1 < 1/2, \alpha_2 < 1/2$, (S.6) and (S.9) hold immediately. The proof of Lemma S.7 is finished. $\square$

Lemma S.8. *Under Assumptions (A1), (B2), if* $\max_{1\le k<\infty}\mathbb{E}\left|\xi_{1k}\right|^{r_1}<\infty$ *for some* $r_1>6$*, then*

$$\max_{1\le k<k'\le\kappa_n}\left|n^{-1}\sum_{i=1}^{n}\xi_{ik}\xi_{ik'}\right|=\mathcal{O}_{a.s.}\left(n^{-1/2}\log^{1/2}n\right), \tag{S.10}$$

$$\max_{1\le k\le\kappa_n}\left|n^{-1}\sum_{i=1}^{n}\xi_{ik}^2-1\right|=\mathcal{O}_{a.s.}\left(n^{-1/2}\log^{1/2}n\right), \tag{S.11}$$

$$\max_{1\le k\le\kappa_n}\left|\bar{\xi}_{\cdot k}\right|=\max_{1\le k\le\kappa_n}\left|n^{-1}\sum_{i=1}^{n}\xi_{ik}\right|=\mathcal{O}_{a.s.}\left(n^{-1/2}\log^{1/2}n\right), \tag{S.12}$$

$$\max_{1\le k\le\kappa_n}\left|n^{-1}\sum_{i=1}^{n}\left|\xi_{ik}\right|-\mu_k\right|=\mathcal{O}_{a.s.}\left(n^{-1/2}\log^{1/2}n\right), \tag{S.13}$$

where $\mu_k=\mathbb{E}\left|\xi_{ik}\right|\le 1$*.*

Proof. For any $1\le k<k'\le\kappa_n$, denote $\xi_{i,kk'}=\xi_{ik}\xi_{ik'}$ and decompose it into tail part, truncated mean and truncated part $\xi_{i,kk'}=\xi_{i,1,kk'}^{D_n}+\xi_{i,2,kk'}^{D_n}+\mu_{kk'}^{D_n}$,

$$\begin{aligned}\xi_{i,1,kk'}^{D_n} &= \xi_{i,kk'}I_{\left\{\left|\xi_{i,kk'}\right|>D_n\right\}},\mu_{kk'}^{D_n}=\mathbb{E}\left[\xi_{i,kk'}I_{\left\{\left|\xi_{i,kk'}\right|\le D_n\right\}}\right],\\ \xi_{i,2,kk'}^{D_n} &= \xi_{i,kk'}I_{\left\{\left|\xi_{i,kk'}\right|\le D_n\right\}}-\mu_{kk'}^{D_n},\end{aligned}$$

where $D_n=n^{\alpha_1}$ satisfying (S.4)-(S.6) in Lemma S.7.

It is straightforward to verify that $\mu_{kk'}^{D_n} = -\mathbb{E}\left[\xi_{i,kk'} I_{\left\{\left|\xi_{i,kk'}\right| > D_n\right\}}\right]$, hence

$$\begin{aligned}
\max_{1\le k<k'\le \kappa_n} \left|\mu_{kk'}^{D_n}\right| &\le \max_{1\le k<k'\le \kappa_n} \left|\mathbb{E}\left[\xi_{i,kk'} I_{\left\{\left|\xi_{i,kk'}\right| > D_n\right\}}\right]\right| \\
&\le \max_{1\le k<k'\le \kappa_n} \mathbb{E}\xi_{i,kk'}^{r_1} D_n^{-(r_1-1)} \le \left(\max_{1\le k\le \kappa_n} \mathbb{E}\xi_{ik}^{r_1}\right)^2 D_n^{-(r_1-1)},
\end{aligned}$$

consequently the truncated mean is negligible

$$\max_{1\le k<k'\le \kappa_n} \left|n^{-1}\sum_{i=1}^{n} \mu_{kk'}^{D_n}\right| \le \max_{1\le k<k'\le \kappa_n} \left|\mu_{kk'}^{D_n}\right| \le \left(\max_{1\le k\le \kappa_n} \mathbb{E}\xi_{ik}^{r_1}\right)^2 D_n^{-(r_1-1)} = o\left(n^{-1/2}\log^{1/2} n\right) \tag{S.14}$$

by (S.4) that $D_n^{-(r_1-1)} = o\left(n^{-1/2}\log^{1/2} n\right)$ and the assumption that $\max_{1\le k<\infty} \mathbb{E}\left|\xi_{1k}\right|^{r_1} < \infty$.

Next we show that tail part is also negligible. Note that

$$\max_{1\le k<k'\le \kappa_n} \mathbb{P}\left\{\left|\xi_{n,kk'}\right| > D_n\right\} \le \left(\max_{1\le k\le \kappa_n} \mathbb{E}\xi_{nk}^{r_1}\right)^2 D_n^{-r_1},$$

hence by (S.5) that $\sum_{n=1}^{\infty} \kappa_n^2 D_n^{-r_1} < \infty$ and by the assumption that $\max_{1\le k<\infty} \mathbb{E}\left|\xi_{1k}\right|^{r_1} < \infty$,

$$\begin{aligned}
\sum_{n=1}^{\infty} \mathbb{P}\left\{\max_{1\le k<k'\le \kappa_n} \left|\xi_{n,kk'}\right| > D_n\right\} &\le \sum_{n=1}^{\infty} \kappa_n^2 \left(\max_{1\le k\le \kappa_n} \mathbb{E}\xi_{nk}^{r_1}\right)^2 D_n^{-r_1} \\
&\le \left(\max_{1\le k<\infty} \mathbb{E}\xi_{1k}^{r_1}\right)^2 \sum_{n=1}^{\infty} \kappa_n^2 D_n^{-r_1} < \infty.
\end{aligned}$$

Borel-Cantelli lemma then implies that

$$\mathbb{P}\left\{\omega \mid \exists N(\omega), \max_{1\le k<k'\le \kappa_n} \left|\xi_{n,kk'}(\omega)\right| \le D_n \text{ for } n > N(\omega)\right\} = 1,$$

which, because $\{D_n\}_{n=1}^{\infty}$ is monotone increasing, entails that

$$\mathbb{P}\left\{\omega \mid \exists N_1(\omega), \max_{1\le k<k'\le\kappa_n} \left|\xi_{i,kk'}(\omega)\right| \le D_n, 1 \le i \le n \text{ for } n > N_1(\omega)\right\} = 1,$$

hence

$$\mathbb{P}\left\{\omega \mid \exists N_1(\omega), \xi_{i,1,kk'}^{D_n}(\omega) = 0, 1 \le k < k' \le \kappa_n, 1 \le i \le n \text{ for } n > N_1(\omega)\right\} = 1,$$

which further implies that

$$\mathbb{P}\left\{\omega \mid \exists N_1(\omega), \max_{1\le k<k'\le\kappa_n} \left|\xi_{i,1,kk'}^{D_n}(\omega)\right| = 0, 1 \le i \le n \text{ for } n > N_1(\omega)\right\} = 1.$$

One therefore has for any $l > 1/2$,

$$\max_{1\le k<k'\le\kappa_n} \left|n^{-1}\sum_{i=1}^{n} \xi_{i,1,kk'}^{D_n}\right| \le n^{-1}\sum_{i=1}^{n} \max_{1\le k<k'\le\kappa_n} \left|\xi_{i,1,kk'}^{D_n}\right| = \mathcal{O}_{a.s.}\left(n^{-l}\right) = o_{a.s.}\left(n^{-1/2}\log^{1/2} n\right). \tag{S.15}$$

Next, note that $\mathbb{E}\left(\xi_{i,2,kk'}^{D_n}\right) = 0$, one has

$$\operatorname{Var}\left(\xi_{i,2,kk'}^{D_n}\right) = \mathbb{E}\left(\xi_{i,kk'}^2\right) - \mathbb{E}\left(\xi_{i,kk'}^2 I_{\left\{\left|\xi_{i,kk'}\right| > D_n\right\}}\right) - \left(\mathbb{E}\left[\xi_{i,kk'} I_{\left\{\left|\xi_{i,kk'}\right| \le D_n\right\}}\right]\right)^2,$$

thus

$$\max_{1\le k<k'\le\kappa_n} \left|\operatorname{Var}\left(\xi_{i,2,kk'}^{D_n}\right) - 1\right| \le C\left(D_n^{-(r_1-2)} + D_n^{-2(r_1-1)}\right).$$

Notice that

$$\mathbb{E}\left|\xi_{i,2,kk'}^{D_n}\right|^l \le 2^{l-2} D_n^{l-2} \mathbb{E}\left|\xi_{i,2,kk'}^{D_n}\right|^2, l \ge 2,$$

which implies that $\left\{\xi_{i,2,kk'}^{D_n}\right\}_{i=1}^{n}$ satisfies Cramér's condition with constant $2D_n$. By Bernstein inequality in Lemma S.4,

$$\mathbb{P}\left\{\left|\frac{\sum_{i=1}^{n} \xi_{i,2,kk'}^{D_n}}{\sqrt{\sum_{i=1}^{n} \mathbb{E}\left|\xi_{i,2,kk'}^{D_n}\right|^2}}\right| > 8\sqrt{\log n}\right\}$$

$$\leq 2\exp\left\{-\frac{64\log n}{4}\frac{1}{1+8\sqrt{\log n}D_n\left(n\mathbb{E}\left|\xi_{i,2,kk'}^{D_n}\right|^2\right)^{-1/2}}\right\}$$

which by (S.6) that $D_n\sqrt{\log n}n^{-1/2}\to 0$, is bounded for large enough n by

$$2\exp\left\{-16\log n\frac{1}{1+8CD_n\sqrt{\log n}n^{-1/2}}\right\}\leq 2\exp\left\{-\frac{16\log n}{2}\right\}\leq 2n^{-8}.$$

Hence, for $n>N_0$ where N_0 is a large enough constant,

$$\mathbb{P}\left\{\max_{1\leq k<k'\leq\kappa_n}\left|n^{-1}\sum_{i=1}^n\xi_{i,2,kk'}^{D_n}\right|>8\sqrt{n^{-1}\log n}\right\}$$

$$\leq\sum_{1\leq k<k'\leq\kappa_n}\mathbb{P}\left\{\left|n^{-1}\sum_{i=1}^n\xi_{i,2,kk'}^{D_n}\right|>8\sqrt{n^{-1}\log n}\right\}\leq 2n^{-8}\kappa_n^2\leq n^{-6},$$

by (S.2) in Lemma S.7. Then

$$\sum_{n=N_0}^{\infty}\mathbb{P}\left\{\max_{1\leq k<k'\leq\kappa_n}\left|n^{-1}\sum_{i=1}^n\xi_{i,2,kk'}^{D_n}\right|>8\sqrt{n^{-1}\log n}\right\}\leq\sum_{n=N_0}^{\infty}n^{-6}<\infty.$$

Borel-Cantelli's lemma entails that

$$\max_{1\leq k<k'\leq\kappa_n}\left|n^{-1}\sum_{i=1}^n\xi_{i,2,kk'}^{D_n}\right|=\mathcal{O}_{a.s.}\left(n^{-1/2}\log^{1/2}n\right).\tag{S.16}$$

According to (S.14), (S.15) and (S.16), putting the truncated mean, the tail and truncated parts together leads to $\max_{1\leq k<k'\leq\kappa_n}\left|n^{-1}\sum_{i=1}^n\xi_{ik}\xi_{ik'}\right|=\mathcal{O}_{a.s.}\left(n^{-1/2}\log^{1/2}n\right)$. The proof of (S.10) is complete.

For any $1\leq k\leq\kappa_n$, decompose ξ_{ik}^2 as $\xi_{ik}^2=\xi_{ik,1}^{D_n'}+\xi_{ik,2}^{D_n'}+\mu_k^{D_n'}$,

$$\begin{aligned}\xi_{ik,1}^{D_n'}&=\xi_{ik}^2I_{\left\{\left|\xi_{ik}^2\right|>D_n'\right\}},\mu_k^{D_n'}=\mathbb{E}\left[\xi_{ik}^2I_{\left\{\left|\xi_{ik}^2\right|\leq D_n'\right\}}\right],\\ \xi_{ik,2}^{D_n'}&=\xi_{ik}^2I_{\left\{\left|\xi_{ik}^2\right|\leq D_n'\right\}}-\mu_k^{D_n'},\end{aligned}$$

where $D'_n = n^{\alpha_2}$ satisfying (S.7)-(S.9) in Lemma S.7.

Following a similar procedure as shown in (S.14),

$$\max_{1\le k\le\kappa_n}\left|n^{-1}\sum_{i=1}^{n}\mu_k^{D'_n}\right| \le \max_{1\le k\le\kappa_n}\left|\mu_k^{D'_n}\right| \le \max_{1\le k\le\kappa_n}\mathbb{E}\xi_{ik}^{r_1}D_n'^{-(r_1/2-1)} = o\left(n^{-1/2}\log^{1/2}n\right) \tag{S.17}$$

by (S.7) that $D_n'^{-(r_1/2-1)} = o\left(n^{-1/2}\log^{1/2}n\right)$ and the assumption that $\max_{1\le k<\infty}\mathbb{E}\left|\xi_{1k}\right|^{r_1}<\infty$.

For the tail part, note that

$$\max_{1\le k\le\kappa_n}\mathbb{P}\left\{\left|\xi_{nk}^2\right| > D'_n\right\} \le \max_{1\le k\le\kappa_n}\mathbb{E}\xi_{nk}^{r_1}D_n'^{-r_1/2},$$

hence

$$\sum_{n=1}^{\infty}\mathbb{P}\left\{\max_{1\le k\le\kappa_n}\left|\xi_{nk}^2\right| > D'_n\right\} \le \sum_{n=1}^{\infty}\kappa_n^2\max_{1\le k\le\kappa_n}\mathbb{E}\xi_{nk}^{r_1}D_n'^{-r_1/2}<\infty,$$

by (S.8) that $\sum_{n=1}^{\infty}\kappa_n^2 D_n'^{-r_1/2}<\infty$ and by the assumption that $\max_{1\le k<\infty}\mathbb{E}\left|\xi_{1k}\right|^{r_1}<\infty$. Borel-Cantelli lemma and D'_n's monotone increasing property entails that

$$\mathbb{P}\left\{\omega \mid \exists N(\omega), \max_{1\le k\le\kappa_n}\left|\xi_{nk}^2(\omega)\right| \le D'_n \text{ for } n > N(\omega)\right\} = 1,$$

$$\mathbb{P}\left\{\omega \mid \exists N_1(\omega), \max_{1\le k\le\kappa_n}\left|\xi_{nk}^2(\omega)\right| \le D'_n, 1\le i\le n \text{ for } n > N_1(\omega)\right\} = 1.$$

Then similar to the proof of (S.15), one obtains that for any $l > 1/2$,

$$\max_{1\le k\le\kappa_n}\left|n^{-1}\sum_{i=1}^{n}\xi_{ik,1}^{D'_n}\right| \le n^{-1}\sum_{i=1}^{n}\max_{1\le k\le\kappa_n}\left|\xi_{ik,1}^{D'_n}\right| = \mathcal{O}_{a.s.}\left(n^{-l}\right) = o_{a.s.}\left(n^{-1/2}\log^{1/2}n\right). \tag{S.18}$$

As for the truncated part, note that $\mathbb{E}\left(\xi_{ik,2}^{D_n'}\right)=0$,

$$\operatorname{Var}\left(\xi_{ik,2}^{D_n'}\right)=\mathbb{E}\left(\xi_{ik}^4\right)-\mathbb{E}\left(\xi_{ik}^4 I_{\left\{\left|\xi_{ik}^2\right|>D_n'\right\}}\right)-\left(\mathbb{E}\left[\xi_{ik}^2 I_{\left\{\left|\xi_{ik}^2\right|\le D_n'\right\}}\right]\right)^2,$$

thus

$$\max_{1\le k\le\kappa_n}\left|\operatorname{Var}\left(\xi_{ik,2}^{D_n'}\right)-\mathbb{E}\xi_{ik}^4\right|\le C\left(D_n'^{-(r_1/2-2)}+D_n'^{-(r_1-2)}\right).$$

Notice that

$$\mathbb{E}\left|u_{i,2,kk'}^{D_n}\right|^l\le 2^{l-2}D_n^{l-2}\mathbb{E}\left|u_{i,2,kk'}^{D_n}\right|^2, l\ge 2,$$

which implies that $\left\{\xi_{ik,2}^{D_n'}\right\}_{i=1}^n$ satisfies Cramér's condition with constant $2D_n'$. By Bernstein inequality in Lemma S.4,

$$\mathbb{P}\left\{\left|\frac{\sum_{i=1}^n \xi_{ik,2}^{D_n'}}{\sqrt{\sum_{i=1}^n \mathbb{E}\left|\xi_{ik,2}^{D_n'}\right|^2}}\right|>8\sqrt{\log n}\right\}$$

$$\le 2\exp\left\{-\frac{64\log n}{4}\frac{1}{1+8\sqrt{\log n}D_n'\left(n\mathbb{E}\left|\xi_{ik,2}^{D_n'}\right|^2\right)^{-1/2}}\right\}$$

which by (S.9) that $D_n'\sqrt{\log n}n^{-1/2}\to 0$, is bounded by

$$2\exp\left\{-16\log n\frac{1}{1+8C\left(\mathbb{E}\xi_{ik}^4\right)^{-1/2}D_n'\sqrt{\log n}n^{-1/2}}\right\}\le 2\exp\left\{-\frac{16\log n}{2}\right\}\le 2n^{-8}$$

for $n>N_0$ where N_0 is a large enough constant. Hence, according to (S.2),

$$\sum_{n=N_0}^{\infty}\mathbb{P}\left\{\max_{1\le k\le\kappa_n}\left|n^{-1}\sum_{i=1}^n\xi_{ik,2}^{D_n'}\right|>8\sqrt{n^{-1}\log n}\right\}$$

$$\le\sum_{n=N_0}^{\infty}\sum_{1\le k\le\kappa_n}\mathbb{P}\left\{\left|n^{-1}\sum_{i=1}^n\xi_{ik,2}^{D_n'}\right|>8\sqrt{n^{-1}\log n}\right\}\le\sum_{n=N_0}^{\infty}n^{-7}<\infty.$$

Borel-Cantelli's lemma entails that

$$\max_{1\le k\le \kappa_n}\left|n^{-1}\sum_{i=1}^{n}\xi_{ik,2}^{D_n'}\right| = \mathcal{O}_{a.s.}\left(n^{-1/2}\log^{1/2}n\right). \tag{S.19}$$

The above (S.17), (S.18) and (S.19) together imply that $\max_{1\le k<k'\le\kappa_n}\left|n^{-1}\sum_{i=1}^{n}\xi_{ik}\xi_{ik'}\right| = \mathcal{O}_{a.s.}\left(n^{-1/2}\log^{1/2}n\right)$. The proof of (S.11) is complete.

(S.12) and (S.13) can be proved in a similar way. □

Proposition 1. *Under Assumptions (A1), (B1)-(B2) and under H_0 in (2.13), as $n\to\infty$,*

$$\max_{1\le i\le n, 1\le k<\infty}\left|\hat{\zeta}_{ik}-\zeta_{ik}+\bar{\zeta}_{\cdot k}\right| = \mathcal{O}_{a.s.}\left(\rho_{n,N}\right), \tag{S.20}$$

$$\max_{1\le k<\infty}\left|\bar{\zeta}_{\cdot k}\right| = \mathcal{O}_{p}\left(n^{-1/2}\right), \tag{S.21}$$

$$\max_{1\le i\le n, 1\le k<\infty}\left|\hat{\zeta}_{ik}-\zeta_{ik}\right| = \mathcal{O}_{p}\left(n^{-1/2}\right), \tag{S.22}$$

where $\bar{\zeta}_{\cdot k} = n^{-1}\sum_{i=1}^{n}\zeta_{ik}$, and $\rho_{n,N}$ in (2.35) of Assumption (B2).

Proof. By definition of $\hat{\zeta}_{ik}$ in (1.6), under H_0

$$\max_{1\le i\le n, 1\le k<\infty}\left|\hat{\zeta}_{ik}-\zeta_{ik}+\bar{\zeta}_{\cdot k}\right| = \max_{1\le i\le n, 1\le k<\infty}\left|\int\left\{\hat{\xi}_i(x)-\xi_i(x)+n^{-1}\sum_{i'=1}^{n}\xi_{i'}(x)\right\}\psi_{0,k}(x)\,dx\right|$$

$$
\begin{aligned}
&\leq \max_{1\leq i\leq n}\left\|\hat{\xi}_i-\xi_i+n^{-1}\sum_{i'=1}^{n}\xi_{i'}\right\|_\infty \max_{1\leq k<\infty}\left\|\psi_{0,k}\right\|_1 \\
&\leq \max_{1\leq i\leq n}\left\|\hat{\xi}_i-\xi_i+n^{-1}\sum_{i'=1}^{n}\xi_{i'}\right\|_\infty \max_{1\leq k<\infty}\left\|\psi_{0,k}\right\|_2 \\
&= \max_{1\leq i\leq n}\left\|\hat{\xi}_i-\xi_i+n^{-1}\sum_{i'=1}^{n}\xi_{i'}\right\|_\infty = \mathcal{O}_{a.s.}\left(\rho_{n,N}\right) = o_{a.s.}\left(n^{-1/2}\kappa_n^{-2}\log^{-1/2}n\right),
\end{aligned}
$$

according to Assumption (B2), so (S.20) is proved.

Assumptions (A1) and (B1) lead to Lemma S.6, which, together with the following arguments establish (S.21)

$$
\begin{aligned}
\max_{1\leq k<\infty}\left|\bar{\zeta}_{\cdot k}\right| &= \max_{1\leq k<\infty}\left|\int\left\{n^{-1}\sum_{i'=1}^{n}\xi_{i'}\left(x\right)\right\}\psi_{0,k}\left(x\right)dx\right| \\
&\leq \left\|n^{-1}\sum_{i=1}^{n}\xi_i\right\|_\infty \max_{1\leq k<\infty}\left\|\psi_{0,k}\right\|_1 \\
&\leq \left\|n^{-1}\sum_{i=1}^{n}\xi_i\right\|_\infty \max_{1\leq k<\infty}\left\|\psi_{0,k}\right\|_2 = \left\|n^{-1}\sum_{i=1}^{n}\xi_i\right\|_\infty = \mathcal{O}_p\left(n^{-1/2}\right).
\end{aligned}
$$

Putting together (S.20) and (S.21) yields (S.22), namely

$$
\max_{1\leq i\leq n, 1\leq k<\infty}\left|\hat{\zeta}_{ik}-\zeta_{ik}\right| = \mathcal{O}_p\left(\rho_{n,N}+n^{-1/2}\right) = \mathcal{O}_p\left(n^{-1/2}\right).
$$

The proof is completed. □

Proposition 2. *Under Assumptions (A1), (B1)-(B2), as $n\to\infty$,*

$$
\sup_{k\in\mathbb{N}_+}\hat{\lambda}_k \leq \Lambda_n = 2n^{-1}\sum_{i=1}^{n}\int\xi_i^2\left(x\right)dx + 2\left(\max_{1\leq i\leq n}\left\|\hat{\xi}_i-\xi_i\right\|_\infty\right)^2 = \mathcal{O}_p\left(1\right). \tag{S.23}
$$

$\left\{\hat{Z}_{kk'}\right\}_{1\leq k,k'<\infty}$ *approximate* $\left\{Z_{kk'}\right\}_{1\leq k,k'<\infty}$ *uniformly:*

$$\max_{1\leq k,k'<\infty}\left|\hat{Z}_{kk'}-Z_{kk'}\right|=\mathcal{O}_p\left(\rho_{n,N}+n^{-1}\right)=o_p\left(n^{-1/2}\kappa_n^{-2}\log^{-1/2}n\right). \tag{S.24}$$

Under H_0 *in (2.13), let* $\bar{\lambda}_k=Z_{kk}=n^{-1}\sum_{i=1}^n\zeta_{ik}^2$*, as* $n\to\infty$*,*

$$\max_{1\leq k<\infty}\left|\hat{\lambda}_k-\bar{\lambda}_k\right| = \mathcal{O}_p\left(\rho_{n,N}+n^{-1}\right), \tag{S.25}$$

$$\max_{1\leq k\leq\kappa_n}\left|\bar{\lambda}_k-\lambda_k\right| = \mathcal{O}_{a.s}\left(n^{-1/2}\log^{1/2}n\right), \tag{S.26}$$

$$\max_{1\leq k\leq\kappa_n}\left|\hat{\lambda}_k-\lambda_k\right| = \mathcal{O}_p\left(n^{-1/2}\log^{1/2}n\right). \tag{S.27}$$

Proof. By definition in (1.7), one can write

$$\hat{\lambda}_k=n^{-1}\sum_{i=1}^n\hat{\zeta}_{ik}^2=n^{-1}\sum_{i=1}^n\left\{\int\hat{\xi}_i(x)\,\psi_{0,k}(x)\,dx\right\}^2$$

$$=n^{-1}\sum_{i=1}^n\left\{\int\left\{\xi_i(x)+\hat{\xi}_i(x)-\xi_i(x)\right\}\psi_{0,k}(x)\,dx\right\}^2$$

$$\leq 2n^{-1}\sum_{i=1}^n\left\{\int\xi_i(x)\,\psi_{0,k}(x)\,dx\right\}^2+2n^{-1}\sum_{i=1}^n\left\{\int\left\{\hat{\xi}_i(x)-\xi_i(x)\right\}\psi_{0,k}(x)\,dx\right\}^2$$

$$\leq 2n^{-1}\sum_{i=1}^n\left\|\xi_i\right\|_2^2\left\|\psi_{0,k}(x)\right\|_2^2+2n^{-1}\sum_{i=1}^n\left\|\hat{\xi}_i-\xi_i\right\|_2^2\left\|\psi_{0,k}(x)\right\|_2^2$$

$$\leq 2n^{-1}\sum_{i=1}^n\int\xi_i^2(x)\,dx+2n^{-1}\sum_{i=1}^n\left\|\hat{\xi}_i-\xi_i\right\|_\infty^2.$$

Thus,

$$\hat{\lambda}_k\leq\Lambda_n=2n^{-1}\sum_{i=1}^n\int\xi_i^2(x)\,dx+2\left(\max_{1\leq i\leq n}\left\|\hat{\xi}_i-\xi_i\right\|_\infty\right)^2$$

since $\max_{1\leq i\leq n}\left\|\hat{\xi}_i-\xi_i\right\|_\infty=\mathcal{O}_p\left(n^{-1/2}\right)$ by (S.1) while $n^{-1}\sum_{i=1}^n\int\xi_i^2\left(x\right)dx\overset{a.s.}{\to}\int G\left(x,x\right)dx<\infty$ by strong law of large numbers, thus $\Lambda_n=\mathcal{O}_p\left(1\right)$ and (S.23) is proved.

Note that

$$\hat{Z}_{kk'}=n^{-1}\sum_{i=1}^n\int\hat{\xi}_i\left(x\right)\psi_{0,k}\left(x\right)dx\int\hat{\xi}_i\left(x'\right)\psi_{0,k'}\left(x'\right)dx',$$

thus $\hat{Z}_{kk'}-Z_{kk'}$ is decomposed as

$$\hat{Z}_{kk'}-Z_{kk'}=I_1+I_2+I_3,$$

$$I_1=n^{-1}\sum_{i=1}^n\int\left\{\hat{\xi}_i\left(x\right)-\xi_i\left(x\right)\right\}\psi_{0,k}\left(x\right)dx\int\xi_i\left(x\right)\psi_{0,k'}\left(x\right)dx,$$

$$I_2=n^{-1}\sum_{i=1}^n\int\left\{\hat{\xi}_i\left(x\right)-\xi_i\left(x\right)\right\}\psi_{0,k'}\left(x\right)dx\int\xi_i\left(x\right)\psi_{0,k}\left(x\right)dx,$$

$$I_3=n^{-1}\sum_{i=1}^n\int\left\{\hat{\xi}_i\left(x\right)-\xi_i\left(x\right)\right\}\psi_{0,k}\left(x\right)dx\int\left\{\hat{\xi}_i\left(x\right)-\xi_i\left(x\right)\right\}\psi_{0,k'}\left(x\right)dx.$$

Note that maximum of the third term is $\max_{1\leq k,k'<\infty}\left|I_3\right|$ is bounded as

$$\max_{1\leq k,k'<\infty}\left|n^{-1}\sum_{i=1}^n\int\left\{\hat{\xi}_i\left(x\right)-\xi_i\left(x\right)\right\}\psi_{0,k}\left(x\right)dx\int\left\{\hat{\xi}_i\left(x\right)-\xi_i\left(x\right)\right\}\psi_{0,k'}\left(x\right)dx\right|$$

$$\leq\left(\max_{1\leq i\leq n}\left\|\hat{\xi}_i-\xi_i\right\|_\infty\right)^2\max_{1\leq k,k'<\infty}\int\left|\psi_{0,k}\left(x\right)\right|dx\int\left|\psi_{0,k'}\left(x\right)\right|dx$$

$$\leq\left(\max_{1\leq i\leq n}\left\|\hat{\xi}_i-\xi_i\right\|_\infty\right)^2\max_{1\leq k<\infty}\int\psi_{0,k}^2\left(x\right)dx$$

$$=\left(\max_{1\leq i\leq n}\left\|\hat{\xi}_i-\xi_i\right\|_\infty\right)^2=\mathcal{O}_p\left(n^{-1}\right).$$

Maximum of the first term $\max_{1\le k,k'<\infty} |I_1|$ is bounded as

$$\begin{aligned}
&\max_{1\le k,k'<\infty}\left|n^{-1}\sum_{i=1}^{n}\int\left\{\hat{\xi}_i(x)-\xi_i(x)\right\}\psi_{0,k}(x)\,dx\int\xi_i(x)\,\psi_{0,k'}(x)\,dx\right| \\
= &\max_{1\le k,k'<\infty}\left|n^{-1}\sum_{i=1}^{n}\int\left\{\hat{\xi}_i(x)-\xi_i(x)+n^{-1}\sum_{i'=1}^{n}\xi_{i'}(x)\right\}\psi_{0,k}(x)\,dx\int\xi_i(x)\,\psi_{0,k'}(x)\,dx\right. \\
&\left.-\int n^{-1}\sum_{i'=1}^{n}\xi_{i'}(x)\,\psi_{0,k}(x)\,dx\int n^{-1}\sum_{i=1}^{n}\xi_i(x)\,\psi_{0,k'}(x)\,dx\right| \\
\le &\max_{1\le k,k'<\infty}\left\|\hat{\xi}_i-\xi_i+n^{-1}\sum_{i'=1}^{n}\xi_{i'}\right\|_\infty\int\left|\psi_{0,k}(x)\right|dx\int n^{-1}\sum_{i=1}^{n}\left|\xi_i(x)\right|\left|\psi_{0,k'}(x)\right|dx \\
&+\max_{1\le k,k'<\infty}\int\left|n^{-1}\sum_{i'=1}^{n}\xi_{i'}(x)\right|\left|\psi_{0,k}(x)\right|dx\int\left|n^{-1}\sum_{i=1}^{n}\xi_i(x)\right|\left|\psi_{0,k'}(x)\right|dx \\
&=\mathcal{O}_{a.s.}\left(\rho_{n,N}\right)\times\mathcal{O}_p(1)+\mathcal{O}_p\left(n^{-1/2}\times n^{-1/2}\right)=\mathcal{O}_p\left(\rho_{n,N}+n^{-1}\right).
\end{aligned}$$

The same bound is good for $\max_{1\le k,k'<\infty} |I_2|$, thus one has established that

$$\max_{1\le k,k'<\infty}\left|\hat{Z}_{kk'}-Z_{kk'}\right|=\mathcal{O}_p\left(\rho_{n,N}+n^{-1}\right).$$

According to (1.5), $\lambda_k = \mathbb{E}\zeta_{ik}^2$. Denote $\bar{\lambda}_k = Z_{kk} = n^{-1}\sum_{i=1}^{n}\zeta_{ik}^2$. Applying (S.11) in Lemma S.8, one easily obtains that

$$\begin{aligned}
\max_{1\le k\le\kappa_n}\left|\bar{\lambda}_k-\lambda_k\right| &\le \max_{1\le k\le\kappa_n}\left|n^{-1}\sum_{i=1}^{n}\zeta_{ik}^2-\lambda_k\right|\le\lambda_1\max_{1\le k\le\kappa_n}\left|n^{-1}\sum_{i=1}^{n}\xi_{ik}^2-1\right| \\
&= \mathcal{O}_{a.s.}\left(n^{-1/2}\log^{1/2}n\right),
\end{aligned}$$

which is (S.26). (S.25) follows as a corollary of (S.24), and (S.27) follows from (S.25) and (S.26). The proof is completed. □

S.2 Results on B-spline Estimates

Matrix algebra represents the B-spline estimator $\hat{\eta}_i(\cdot)$ in (2.38) as

$$\hat{\eta}_i(x) = \{B_{1,p}(x), \ldots, B_{J_s+p,p}(x)\} \left(\mathbf{B}^\mathtt{T}\mathbf{B}\right)^{-1} \mathbf{B}^\mathtt{T}\mathbf{Y}_i,$$

where $\mathbf{Y}_i = (Y_{i1}, \ldots, Y_{iN})^\mathtt{T}$ and the design matrix $\mathbf{B}$ is

$$\mathbf{B} = \begin{pmatrix} B_{1,p}(1/N) & \cdots & B_{J_s+p,p}(1/N) \\ \vdots & \cdots & \vdots \\ B_{1,p}(N/N) & \cdots & B_{J_s+p,p}(N/N) \end{pmatrix}_{N\times(J_s+p)} = \{\mathbf{B}(1/N), \ldots, \mathbf{B}(N/N)\}^\mathtt{T},$$

in which $\mathbf{B}(x)^\mathtt{T} = \{B_{1,p}(x), \ldots, B_{J_s+p,p}(x)\}$. Denote the theoretical and empirical inner product matrixs of B-spline basis $\{B_{J,p}(x)\}_{J=1}^{J_s+p}$ as

$$\mathbf{V}_p = \{\langle B_{J,p}, B_{J',p}\rangle\}_{J,J'=1}^{J_s+p}, \tag{S.28}$$

$$\mathbf{V}_{n,p} = \left\{\langle B_{J,p}, B_{J',p}\rangle_N\right\}_{J,J'=1}^{J_s+p} = N^{-1}\mathbf{B}^\mathtt{T}\mathbf{B}. \tag{S.29}$$

According to model (1.2), one obtains

$$\boldsymbol{\eta}_i = \mathbf{m} + \boldsymbol{\xi}_i,$$

where $\boldsymbol{\eta}_i = \{\eta_i(1/N), \ldots, \eta_i(N/N)\}^\mathtt{T}$, $\mathbf{m} = \{m(1/N), \ldots, m(N/N)\}^\mathtt{T}$ and

$\boldsymbol{\xi}_i = \{\xi_i(1/N), \ldots, \xi_i(N/N)\}^\mathtt{T}$. Then, denote $\boldsymbol{\varepsilon}_i = (\sigma_i(1/N)\varepsilon_{i1}, \ldots, \sigma_i(N/N)\varepsilon_{iN})^\mathtt{T}$,

$\widehat{\eta}_i(x)$ can be decomposed as:

$$\hat{\eta}_i(x) = \tilde{\eta}_i(x) + \tilde{\varepsilon}_i(x),$$

where

$$\begin{aligned}
\tilde{\eta}_i(x) &= N^{-1}\mathbf{B}(x)^{\intercal}\mathbf{V}_{n,p}^{-1}\mathbf{B}^{\intercal}\boldsymbol{\eta}_i = \tilde{m}(x) + \tilde{\xi}_i(x), \\
\tilde{m}(x) &= N^{-1}\mathbf{B}(x)^{\intercal}\mathbf{V}_{n,p}^{-1}\mathbf{B}^{\intercal}\mathbf{m}, \\
\tilde{\xi}_i(x) &= N^{-1}\mathbf{B}(x)^{\intercal}\mathbf{V}_{n,p}^{-1}\mathbf{B}^{\intercal}\boldsymbol{\xi}_i, \qquad \text{(S.30)} \\
\tilde{\varepsilon}_i(x) &= N^{-1}\mathbf{B}(x)^{\intercal}\mathbf{V}_{n,p}^{-1}\mathbf{B}^{\intercal}\boldsymbol{\varepsilon}_i. \qquad \text{(S.31)}
\end{aligned}$$

So $\hat{\eta}_i(x) = \tilde{\xi}_i(x) + \tilde{m}(x) + \tilde{\varepsilon}_i(x)$. By the definition of $\hat{\xi}_i(x), \hat{m}(x)$ in (2.40) and (2.39)

$$\begin{aligned}
&\hat{\xi}_i(x) - \xi_i(x) = \hat{\eta}_i(x) - \hat{m}(x) - \xi_i(x) \\
&= \tilde{\xi}_i(x) + \tilde{m}(x) + \tilde{\varepsilon}_i(x) - n^{-1}\sum_{i'=1}^{n}\left\{\tilde{\xi}_i(x) + \tilde{m}(x) + \tilde{\varepsilon}_i(x)\right\} - \xi_i(x) \\
&= \tilde{\xi}_i(x) - \xi_i(x) + \tilde{\varepsilon}_i(x) - n^{-1}\sum_{i'=1}^{n}\left\{\tilde{\xi}_{i'}(x) - \xi_{i'}(x) + \tilde{\varepsilon}_{i'}(x)\right\} - n^{-1}\sum_{i'=1}^{n}\xi_{i'}(x).
\end{aligned} \qquad \text{(S.32)}$$

Lemma S.9. *There exist $0 < c_p < C_p < \infty$ such that*

$$c_p \left\|\boldsymbol{\alpha}\right\|_2 J_s^{-1/2} \leq \left\|\sum_{J=1}^{J_s+p}\alpha_J B_{J,p}\right\|_2 \leq C_p \left\|\boldsymbol{\alpha}\right\|_2 J_s^{-1/2}, \forall \boldsymbol{\alpha} = (\alpha_1, \ldots \alpha_{J_s+p}) \in \mathbb{R}^{J_s+p}, \qquad \text{(S.33)}$$

and the theoretical inner product matrix $\mathbf{V}_p$ *in (S.28) satisfies*

$$c_p J_s \|\boldsymbol{\alpha}\|_\infty \leq \left\|\mathbf{V}_p^{-1}\boldsymbol{\alpha}\right\|_\infty \leq C_p J_s \|\boldsymbol{\alpha}\|_\infty, \forall \boldsymbol{\alpha} = (\alpha_1, \ldots \alpha_{J_s+p}) \in \mathbb{R}^{J_s+p}. \tag{S.34}$$

Under Assumption (C5), for n *large enough, the empirical inner product matrix* $\mathbf{V}_{n,p}$ *in (S.29) satisfies*

$$c_p J_s \|\boldsymbol{\alpha}\|_\infty \leq \left\|\mathbf{V}_{n,p}^{-1}\boldsymbol{\alpha}\right\|_\infty \leq C_p J_s \|\boldsymbol{\alpha}\|_\infty, \forall \boldsymbol{\alpha} = (\alpha_1, \ldots \alpha_{J_s+p}) \in \mathbb{R}^{J_s+p}. \tag{S.35}$$

Proof. (S.33) follows from Theorem 5.4.2 of DeVore and Lorentz (1993), while (S.34), (S.35) follow from Lemma A.3 and Lemma A.4 of Cao et al. (2012). □

Lemma S.10. *Assumptions (C3) and (C4') imply Assumption (C4).*

Proof. Taking $H(x) = x^{r_2}$, Lemma S.1 entails that there exist constants c and a depending on the distribution of ε_{ij}, such that for $x_N = N^{\beta_2}$, $N/H(ax_N) = a^{-r_2}N^{1-r_2\beta_2}$ and i.i.d. standard normal random variables $\{U_{ij,\varepsilon}\}_{i=1,j=1}^{n,N}$ such that

$$\max_{1\leq i\leq n} \mathbb{P}\left\{\max_{1\leq t\leq N}\left|\sum_{j=1}^{t}\varepsilon_{ij} - \sum_{j=1}^{t}U_{ij,\varepsilon}\right| > N^{\beta_2}\right\} < ca^{-r_2}N^{1-r_2\beta_2}.$$

By Assumption (C3) that $n = \mathcal{O}\left(N^\theta\right)$, there exists a constant $C_\varepsilon > 0$ such that

$$\mathbb{P}\left\{\max_{1\le i\le n}\max_{1\le t\le N}\left|\sum_{j=1}^{t}\varepsilon_{ij}-\sum_{j=1}^{t}U_{ij,\varepsilon}\right|>N^{\beta_2}\right\}<ca^{-r_2}n\times N^{1-r_2\beta_2}\le C_\varepsilon N^{\theta+1-r_2\beta_2}.$$

Since $r_2>(2+\theta)/\beta_2$ according to Assumption (C4'), there is $\gamma_2=r_2\beta_2-1-\theta>1$ and Assumption (C4) follows. □

Under Assumption (C5) and (2.45), $J_s=N^{\gamma}d_N$ with $d_N+d_N^{-1}=\mathcal{O}\left(\log^{\tau}N\right)$ and $\gamma>1-\nu$. Thus, $J_sN^{-(1-\nu)}=N^{\gamma-(1-\nu)}d_N\to\infty$ as $N\to\infty$, which leads to

$$J_sN^{-1}\gg N^{-\nu}. \tag{S.36}$$

Lemma S.11. *Under Assumptions (C1), (C4) and (C5), as $n\to\infty$*

$$\max_{1\le i\le n}\left\|N^{-1}\mathbf{B}^{\intercal}\boldsymbol{\varepsilon}_i\right\|_\infty=\mathcal{O}_{a.s.}\left(N^{-1/2}J_s^{-1/2}\log^{1/2}N+N^{\beta_2-1}\right),$$

$$\max_{1\le i\le n}\left\|\tilde{\varepsilon}_i\right\|_\infty=\mathcal{O}_{a.s.}\left(N^{-1/2}J_s^{1/2}\log^{1/2}N+J_sN^{\beta_2-1}\right). \tag{S.37}$$

Proof. According to Assumption (C4), it is easy to see

$$\max_{1\le i\le n}\max_{1\le t\le N}\left|N^{-1}\sum_{j=1}^{t}\left(\varepsilon_{ij}-U_{ij,\varepsilon}\right)\right|=\mathcal{O}_{a.s.}(N^{\beta_2-1}).$$

Note that the B-spline basis satisfies

$$\left|B_{J,p}\left(\frac{j}{N}\right)-B_{J,p}\left(\frac{j+1}{N}\right)\right|\le N^{-1}\left\|B_{J,p}\right\|_{0,1}\le CJ_sN^{-1}$$

uniformly over $1\le j\le N$ and $1\le J\le J_s+p$, while (S.36) provides that

$J_s N^{-1} \gg N^{-\nu}$, hence

$$\max_{1\leq i\leq n}\left|\sigma_i\left(\frac{j}{N}\right)-\sigma_i\left(\frac{j+1}{N}\right)\right| \leq N^{-\nu}\max_{1\leq i\leq n}\|\sigma_i\|_{0,\nu} \leq CJ_sN^{-1}$$

uniformly over $1 \leq j \leq N$ by Assumption (C1). Since for $1 \leq J \leq J_s + p$, both $B_{J,p}(\cdot)$ and $\sigma_i(\cdot)$ are bounded on $[0, 1]$, then

$$\begin{aligned}
&\max_{1\leq i\leq n}\left|B_{J,p}\left(\frac{j}{N}\right)\sigma_i\left(\frac{j}{N}\right)-B_{J,p}\left(\frac{j+1}{N}\right)\sigma_i\left(\frac{j+1}{N}\right)\right| \\
&\quad= \max_{1\leq i\leq n}\left|\left\{B_{J,p}\left(\frac{j}{N}\right)-B_{J,p}\left(\frac{j+1}{N}\right)+B_{J,p}\left(\frac{j+1}{N}\right)\right\}\sigma_i\left(\frac{j}{N}\right)-B_{J,p}\left(\frac{j+1}{N}\right)\sigma_i\left(\frac{j+1}{N}\right)\right| \\
&\quad\leq \left|B_{J,p}\left(\frac{j}{N}\right)-B_{J,p}\left(\frac{j+1}{N}\right)\right|\max_{1\leq i\leq n}\sigma_i\left(\frac{j}{N}\right)+\max_{1\leq i\leq n}\left|\sigma_i\left(\frac{j}{N}\right)-\sigma_i\left(\frac{j+1}{N}\right)\right|B_{J,p}\left(\frac{j+1}{N}\right) \\
&\quad\leq CJ_sN^{-1}.
\end{aligned}$$

Since the support of $B_{J,p}(\cdot)$ has length at most $p/\left(J_s+1\right)$, one obtains that

$$\max_{1\leq i\leq n}\sum_{j=1}^{N-1}\left|B_{\ell,p}\left(\frac{j}{N}\right)\sigma_i\left(\frac{j}{N}\right)-B_{\ell,p}\left(\frac{j+1}{N}\right)\sigma_i\left(\frac{j+1}{N}\right)\right| \leq C.$$

Hence,

$$\begin{aligned}
&\max_{1\leq i\leq n}\left|N^{-1}\sum_{j=1}^{N}B_{J,p}(j/N)\sigma_i\left(j/N\right)\left(\varepsilon_{ij}-U_{ij,\varepsilon}\right)\right| \\
&= \max_{1\leq i\leq n}\sum_{j=1}^{N-1}\left\{B_{J,p}\left(\frac{j}{N}\right)\sigma_i\left(\frac{j}{N}\right)-B_{J,p}\left(\frac{j+1}{N}\right)\sigma_i\left(\frac{j+1}{N}\right)\right\}N^{-1}\sum_{t=1}^{j}\left(\varepsilon_{it}-U_{it,\varepsilon}\right) \\
&\quad+\max_{1\leq i\leq n}B_{J,p}\left(1\right)\sigma_i\left(1\right)N^{-1}\sum_{t=1}^{N}\left(\varepsilon_{it}-U_{it,\varepsilon}\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \left\{\max_{1\leq i\leq n}\max_{1\leq j\leq N}\left|N^{-1}\sum_{t=1}^{j}\left(\varepsilon_{it}-U_{it,\varepsilon}\right)\right|\right\}\times \\
&\quad \left\{\sum_{j=1}^{N-1}\left|B_{J,p}\left(\frac{j}{N}\right)\sigma_i\left(\frac{j}{N}\right)-B_{J,p}\left(\frac{j+1}{N}\right)\sigma_i\left(\frac{j+1}{N}\right)\right|+C\right\} \\
&= \mathcal{O}_{a.s}\left(N^{\beta_2-1}\right).
\end{aligned}$$

Thus,

$$\max_{1\leq i\leq n}\max_{1\leq J\leq J_s+p}\left|N^{-1}\sum_{j=1}^{N}B_{J,p}(j/N)\sigma_i\left(j/N\right)\left(\varepsilon_{ij}-U_{ij,\varepsilon}\right)\right|=\mathcal{O}_{a.s}\left(N^{\beta_2-1}\right).$$

According to Lemma A.6 of Wang et al. (2020),

$$\max_{1\leq i\leq n}\max_{1\leq J\leq J_s+p}\left|N^{-1}\sum_{j=1}^{N}B_{J,p}(j/N)\sigma_i\left(j/N\right)U_{ij,\varepsilon}\right|=\mathcal{O}_{a.s.}(N^{-1/2}J_s^{-1/2}\log^{1/2}N).$$

Then, The above inequalities together imply that

$$\begin{aligned}
\max_{1\leq i\leq n}\left\|N^{-1}\mathbf{B}^{\intercal}\boldsymbol{\varepsilon}_i\right\|_\infty &= \max_{1\leq i\leq n}\max_{1\leq J\leq J_s+p}\left|N^{-1}\sum_{j=1}^{N}B_{J,p}(j/N)\sigma_i(j/N)\varepsilon_{ij}\right| \\
&= \mathcal{O}_{a.s.}(N^{-1/2}J_s^{-1/2}\log^{1/2}N+N^{\beta_2-1}).
\end{aligned}$$

Note next that according to (S.31) and (S.35) in Lemma S.9,

$$\max_{1\leq i\leq n}\|\tilde{\varepsilon}_i\|_\infty=\max_{1\leq i\leq n}\sup_{x\in[0,1]}|\tilde{\varepsilon}_i(x)|=\max_{1\leq i\leq n}\sup_{x\in[0,1]}\left|N^{-1}\mathbf{B}(x)^{\intercal}\mathbf{V}_{n,p}^{-1}\mathbf{B}^{\intercal}\boldsymbol{\varepsilon}_i\right|$$

$$=\max_{1\leq i\leq n}\sup_{x\in[0,1]}\left|\mathbf{B}(x)^{\intercal}\mathbf{V}_{n,p}^{-1}N^{-1}\mathbf{B}^{\intercal}\boldsymbol{\varepsilon}_i\right|$$

$$\leq C_pJ_s\max_{1\leq i\leq n}\left\|N^{-1}\mathbf{B}^{\intercal}\boldsymbol{\varepsilon}_i\right\|_\infty=\mathcal{O}_{a.s.}(N^{-1/2}J_s^{1/2}\log^{1/2}N+J_sN^{\beta_2-1}),$$

as for any $x\in[0,1]$, the vector $\mathbf{B}(x)$ has at most p non-zero elements, each taking value in $[0,1]$. So (S.37) is proved, and the proof is completed. □

Lemma S.12. *Under Assumptions (C2) and (C4), as $n \to \infty$,*

$$\max_{1 \le i \le n} \left\| \tilde{\xi}_i - \xi_i \right\|_\infty = \mathcal{O}_{a.s.} \left(J_s^{-p^*} (n \log n)^{2/r_1} \right), \tag{S.38}$$

Under Assumptions (C1)-(C2), (C4)-(C5)

$$\max_{1 \le i \le n} \left\| \hat{\xi}_i - \xi_i + n^{-1} \sum_{i'=1}^{n} \xi_{i'} \right\|_\infty = \mathcal{O}_{a.s.} \left(J_s^{-p^*} (n \log n)^{2/r_1} + N^{-1/2} J_s^{1/2} \log^{1/2} N + J_s N^{\beta_2 - 1} \right). \tag{S.39}$$

Proof. For any $k = 1, 2, \ldots$, let $\boldsymbol{\phi}_k = (\phi_k(1/N), \ldots, \phi_k(N/N))^{\mathrm{T}}$, and denote $\tilde{\phi}_k(x) = N^{-1} \mathbf{B}(x)^{\mathsf{T}} \mathbf{V}_{n,p}^{-1} \mathbf{B}^{\mathsf{T}} \boldsymbol{\phi}_k$. According to (S.30), $\tilde{\xi}_i(x) = \sum_{k=1}^{\infty} \xi_{ik} \tilde{\phi}_k(x)$, therefore,

$$\tilde{\xi}_i(x) - \xi_i(x) = \sum_{k=1}^{\infty} \xi_{ik} \left\{ \tilde{\phi}_k(x) - \phi_k(x) \right\}.$$

By Lemma A.4 of Cao et al. (2012), there exists a constant $C_{q,\mu} > 0$, such that

$$\left\| \tilde{\phi}_k - \phi_k \right\|_\infty \le C_{q,\mu} \left\| \phi_k \right\|_{q,\mu} J_s^{-p^*}, k \ge 1.$$

Thus, one obtains

$$\left\| \tilde{\xi}_i - \xi_i \right\|_\infty \le \sum_{k=1}^{\infty} |\xi_{ik}| \left\| \tilde{\phi}_k - \phi_k \right\|_\infty \le C_{q,\mu} W_i J_s^{-p^*},$$

where $W_i = \sum_{k=1}^{\infty} |\xi_{ik}| \left\| \phi_k \right\|_{q,\mu}, i = 1, \ldots, n$, are i.i.d. nonnegative random variables with finite r_1-th absolute moment. By Assumptions (C2) and (C4), one has

$$\mathbb{P} \left\{ \max_{1 \le i \le n} W_i > (n \log n)^{2/r_1} \right\} \le n \frac{\mathbb{E} W_i^{r_1}}{(n \log n)^2} = \mathbb{E} W_i^{r_1} n^{-1} (\log n)^{-2},$$

thus,

$$\sum_{n=1}^{\infty} \mathbb{P}\left\{\max_{1\leq i\leq n} W_i > (n\log n)^{2/r_1}\right\} \leq \mathbb{E}W_i^{r_1} \sum_{n=1}^{\infty} n^{-1} (\log n)^{-2} < +\infty,$$

so $\max_{1\leq i\leq n} W_i = \mathcal{O}_{a.s.}\left\{(n\log n)^{2/r_1}\right\}$ by Borel-Cantelli Lemma. Thus,

$$\max_{1\leq i\leq n} \left\|\tilde{\xi}_i - \xi_i\right\|_{\infty} = \mathcal{O}_{a.s.}\left(J_s^{-p^*} (n\log n)^{2/r_1}\right),$$

which establishes (S.38).

Putting together Lemma S.6, (S.38), (S.37), and (S.32) yields (S.39). The proof is completed. □

S.3 Proof of Theorem 1

Assumption (A1) and (2.16), (2.17) ensure that

$$\mathbb{E}\left\|\mathbf{X}_i\right\|_{\mathcal{H}}^2 = \sum\nolimits_{k,k'=1}^{\infty} \mathbb{E}\left\{\sum\nolimits_{k_1,k_2=1}^{\infty} \zeta_{ik_1}\zeta_{ik_2}u_{k_1k_2,kk'}\right\}^2$$

$$= \sum\nolimits_{k,k'=1}^{\infty}\sum\nolimits_{k_1,k_2,k_3,k_4=1}^{\infty} \left(\mathbb{E}\zeta_{ik_1}\zeta_{ik_2}\zeta_{ik_3}\zeta_{ik_4}\right) u_{k_1k_2,kk'}u_{k_3k_4,kk'}$$

$$= \sum\nolimits_{k_1,k_2,k_3,k_4=1}^{\infty} \left(\mathbb{E}\zeta_{ik_1}\zeta_{ik_2}\zeta_{ik_3}\zeta_{ik_4}\right) \delta_{k_1k_3}\delta_{k_2k_4}$$

$$= \sum\nolimits_{k_1=1}^{\infty} \mathbb{E}\zeta_{ik_1}^4 + 2\sum\nolimits_{1\leq k_1<k_2<\infty} \mathbb{E}\zeta_{ik_1}^2\zeta_{ik_2}^2$$

$$= \sum\nolimits_{k=1}^{\infty} \lambda_k^2\mathbb{E}\xi_k^4 + 2\sum\nolimits_{1\leq k_1<k_2<\infty} \lambda_{k_1}\lambda_{k_2},$$

$$\mathbb{E}\left\|\mathbf{X}_i\right\|_{\mathcal{H}}^2 = \sum\nolimits_{k=1}^{\infty} \lambda_k^2\left(\mathbb{E}\xi_k^4 - 1\right) + \left(\sum\nolimits_{k=1}^{\infty}\lambda_k\right)^2 < \infty,$$

which establishes (2.27) and that $\|\mathbf{X}_i\|_{\mathcal{H}}^2 < \infty$ almost surely, hence $\{\mathbf{X}_i\}_{i=1}^n$ is an i.i.d. sequence of $\mathcal{H}$-valued random variables according to Bosq (2000), and (2.26) follows also:

$$\boldsymbol{\mu}_{\mathbf{X}} = \mathbb{E}\mathbf{X}_i = \left(\sum\nolimits_{k_1,k_2=1}^{\infty} \mathbb{E}\zeta_{ik_1}\zeta_{ik_2} u_{k_1k_2,kk'}\right)_{1\le k,k'<\infty}$$
$$= \left(\sum\nolimits_{k_1=1}^{\infty} \lambda_{k_1} u_{k_1k_1,kk'}\right)_{1\le k,k'<\infty}.$$

Hilbert space Central Limit Theorem (Theorem 2.7 in Bosq (2000)) then implies (2.28):

$$n^{1/2}\left\{(Z_{kk'})_{1\le k,k'<\infty} - \boldsymbol{\mu}_{\mathbf{X}}\right\} = n^{-1/2}\sum_{i=1}^{n}(\mathbf{X}_i - \boldsymbol{\mu}_{\mathbf{X}}) \xrightarrow{D} N(\mathbf{0}, \mathbf{C}_{\mathbf{X}}).$$

Under Assumption (A1), the definition of $\mathbf{Y}_1$ in (2.23), $\mathbb{E}\mathbf{Y}_1 = \mathbf{0}$, and

$$\mathbb{E}\|\mathbf{Y}_1\|_{\mathcal{H}}^2 = \sum\nolimits_{k_1,k_2=1}^{\infty} \mathbb{E}\left(\zeta_{ik_1}\zeta_{ik_2} - \lambda_{k_1}\delta_{k_1k_2}\right)^2$$
$$= \sum\nolimits_{k_1=1}^{\infty} \mathbb{E}\left(\zeta_{ik_1}^2 - \lambda_{k_1}\right)^2 + \sum\nolimits_{1\le k_1\ne k_2<\infty} \mathbb{E}\zeta_{ik_1}^2\zeta_{ik_2}^2$$
$$= \sum\nolimits_{k_1=1}^{\infty} \lambda_{k_1}^2\left(\mathbb{E}\xi_{ik_1}^4 - 1\right) + \sum\nolimits_{1\le k_1\ne k_2<\infty} \lambda_{k_1}\lambda_{k_2}$$
$$= \sum\nolimits_{k_1=1}^{\infty} \lambda_{k_1}^2\left(\mathbb{E}\xi_{ik_1}^4 - 2\right) + \left(\sum\nolimits_{1\le k<\infty} \lambda_k\right)^2,$$

which equals $\mathbb{E}\|\mathbf{X}_1 - \boldsymbol{\mu}_{\mathbf{X}}\|_{\mathcal{H}}^2$ since operator $\mathbf{U}$ is unitary. According to Bosq (2000), the covariance operator of $\mathbf{Y}_1$ is $\mathbf{C}_{\mathbf{Y}}(\cdot) = \mathbb{E}(\langle \mathbf{Y}_1, \cdot\rangle \mathbf{Y}_1)$, a positive, symmetric and nuclear operator which satisfies (2.30) because

$$\begin{aligned}\mathbf{C}_{\mathbf{Y}}(\mathbf{e}_{kk}) &= \mathbb{E}(\langle \mathbf{Y}_1, \mathbf{e}_{kk}\rangle \mathbf{Y}_1) = \mathbb{E}\left(\zeta_{1k}^2 - \lambda_k\right)\mathbf{Y}_1 \\ &= \mathbb{E}\left(\zeta_{1k}^2 - \lambda_k\right)^2 \mathbf{e}_{kk} = \lambda_k^2\left(\mathbb{E}\xi_{1k}^4 - 1\right)\mathbf{e}_{kk}, 1\le k<\infty\end{aligned}$$

$\mathbf{C}_{\mathbf{Y}}(\mathbf{e}_{kk'}) = \mathbb{E}(\langle \mathbf{Y}_1, \mathbf{e}_{kk'} \rangle \mathbf{Y}_1) = \mathbb{E}\zeta_{1k}\zeta_{1k'}\mathbf{Y}_1 = \lambda_k\lambda_{k'}(\mathbf{e}_{kk'} + \mathbf{e}_{k'k}), 1 \leq k \neq k' < \infty.$

Immediately, (2.29) follows from (2.28) and (2.24) as

$$n^{1/2}(Z_{kk} - \lambda_k, Z_{kk'})_{1 \leq k \neq k' < \infty} = n^{-1/2} \sum_{i=1}^{n} \mathbf{Y}_i \xrightarrow{D} \mathcal{N} \sim N(\mathbf{0}, \mathbf{C}_{\mathbf{Y}}).$$

Define next a function $f : \mathcal{H} \to (0, +\infty)$ as norm square of projection

$$f\left\{(a_{kk'})_{1 \leq k, k' < \infty}\right\} = \sum_{1 \leq k < k' < \infty} a_{kk'}^2 = \left\|\mathcal{P}_{\mathrm{UT}}(a_{kk'})_{1 \leq k, k' < \infty}\right\|_{\mathcal{H}}^2, \qquad \text{(S.40)}$$

which is continuous over $\mathcal{H}$. Applying Banach space Continuous Mapping Theorem (equation (2.11) of Bosq (2000)) to (2.29) and (S.40), one obtains

$$\tilde{S}_n = f\left(n^{-1/2} \sum_{i=1}^{n} \mathbf{Y}_i\right) \xrightarrow{D} f(\mathcal{N}) = \|\mathcal{P}_{\mathrm{UT}}(\mathcal{N})\|_{\mathcal{H}}^2 = \sum_{1 \leq k < k' < \infty} \lambda_k \lambda_{k'} \chi_{kk'}^2(1),$$

where the sum $\sum_{1 \leq k < k' < \infty} \lambda_k \lambda_{k'} \chi_{kk'}^2(1)$ is finite with probability 1 since its expectation $\sum_{1 \leq k < k' < \infty} \lambda_k \lambda_{k'} < \infty$. The proof is completed. □

S.4 Proof of Theorem 2

Denote $\check{S}_n = n \sum_{1 \leq k < k' \leq \kappa_n} Z_{kk'}^2$, then

$$\left|\hat{S}_n - \check{S}_n\right| = n \left|\sum_{1 \leq k < k' \leq \kappa_n} \left(\hat{Z}_{kk'}^2 - Z_{kk'}^2\right)\right| \leq n(\Delta_1 + \Delta_2),$$

$$\Delta_1 = \left|\sum_{1 \leq k < k' \leq \kappa_n} \left(\hat{Z}_{kk'} - Z_{kk'}\right)^2\right|, \Delta_2 = \left|\sum_{1 \leq k < k' \leq \kappa_n} 2Z_{kk'}\left(\hat{Z}_{kk'} - Z_{kk'}\right)\right|. \qquad \text{(S.41)}$$

According to (S.24),

$$\Delta_1 \leq \sum_{1\leq k<k'\leq\kappa_n} \max_{1\leq k,k'\leq\kappa_n} \left|\hat{Z}_{kk'} - Z_{kk'}\right|^2 = \mathcal{O}_p\left(\kappa_n^2\rho_{n,N}^2 + \kappa_n^2 n^{-2}\right). \qquad \text{(S.42)}$$

Notice that

$$\max_{1\leq k<k'\leq\kappa_n} |Z_{kk'}| = \max_{1\leq k<k'\leq\kappa_n} \left|n^{-1}\sum_{i=1}^n \zeta_{ik}\zeta_{ik'}\right| = \max_{1\leq k<k'\leq\kappa_n} \left|n^{-1}\sum_{i=1}^n \sqrt{\lambda_k\lambda_{k'}}\xi_{ik}\xi_{ik'}\right|$$
$$\leq \lambda_1 \max_{1\leq k<k'\leq\kappa_n} \left|n^{-1}\sum_{i=1}^n \xi_{ik}\xi_{ik'}\right| = \mathcal{O}_{a.s.}\left(n^{-1/2}\log^{1/2} n\right)$$

by (S.10) in Lemma S.8. Then

$$\begin{aligned}\Delta_2 &\leq 2\sum_{1\leq k<k'\leq\kappa_n} \max_{1\leq k<k'\leq\kappa_n} |Z_{kk'}| \max_{1\leq k,k'\leq\kappa_n} \left|\hat{Z}_{kk'} - Z_{kk'}\right| \\ &= \kappa_n^2 \times \mathcal{O}_{a.s.}\left(n^{-1/2}\log^{1/2} n\right) \times \mathcal{O}_p\left(\rho_{n,N} + n^{-1}\right) \\ &= \mathcal{O}_p\left(\kappa_n^2 n^{-1/2}\rho_{n,N}\log^{1/2} n + \kappa_n^2 n^{-3/2}\log^{1/2} n\right). \qquad \text{(S.43)}\end{aligned}$$

Now (S.41), (S.42) and (S.43) imply that

$$\begin{aligned}\left|\hat{S}_n - \check{S}_n\right| &= n \times \mathcal{O}_p\left(\kappa_n^2\rho_{n,N}^2 + \kappa_n^2 n^{-2}\right) + \\ &\quad n \times \mathcal{O}_p\left(\kappa_n^2 n^{-1/2}\rho_{n,N}\log^{1/2} n + \kappa_n^2 n^{-3/2}\log^{1/2} n\right) \\ &= \mathcal{O}_p\left(\kappa_n^2 n\rho_{n,N}^2 + \kappa_n^2\rho_{n,N} n^{1/2}\log^{1/2} n + \kappa_n^2 n^{-1/2}\log^{1/2} n\right).\end{aligned}$$

By recalling Assumption (B2) and (2.34) that $\kappa_n^2\rho_{n,N}n^{1/2}\log^{1/2} n \to 0, \kappa_n^2 n^{-1/2}\log^{3/2} n \to 0$ and (S.3) in Lemma S.7, $\left|\hat{S}_n - \check{S}_n\right| = o_p(1)$. On the other hand,

$$\tilde{S}_n - \check{S}_n = \sum_{1\leq k<k',k'>\kappa_n} nZ_{kk'}^2 = n\sum_{1\leq k<k',k'>\kappa_n} \left(n^{-1}\sum_{i=1}^n \zeta_{ik}\zeta_{ik'}\right)^2$$

$$= \sum_{1\le k<k', k'>\kappa_n} \lambda_k \lambda_{k'} \times n^{-1} \left(\sum_{i=1}^{n} \xi_{ik}\xi_{ik'} \right)^2 .$$

Note that under Assumption (A1) for any $1 \le k < k'$,

$$\mathbb{E} n^{-1} \left(\sum_{i=1}^{n} \xi_{ik}\xi_{ik'} \right)^2 = n^{-1} \sum_{i,i'=1}^{n} \mathbb{E} \xi_{ik}\xi_{ik'}\xi_{i'k}\xi_{i'k'} = n^{-1} \sum_{i=1}^{n} \mathbb{E} \xi_{ik}^2 \xi_{ik'}^2 = 1,$$

hence by taking $R_{kk',n} = n^{-1} \left(\sum_{i=1}^{n} \xi_{ik}\xi_{ik'} \right)^2, M = 1$ in Lemma S.5, one obtains that $\tilde{S}_n - \check{S}_n = o_p(1)$. Then,

$$\left| \hat{S}_n - \tilde{S}_n \right| \le \left| \hat{S}_n - \check{S}_n \right| + \left| \check{S}_n - \tilde{S}_n \right| = o_p(1) .$$

Slutsky's Theorem then implies that $\hat{S}_n \xrightarrow{D} S$.

Since S has continuous distribution by Lemma S.2, Pólyà's Theorem (Theorem 11.2.9, Lehmann et al. (2005)) implies that as $n \to \infty$

$$\sup_{q\in\mathbb{R}} \left| \mathbb{P}\left[\tilde{S}_n \le q \right] - \mathbb{P}\left[S \le q \right] \right| \to 0, \sup_{q\in\mathbb{R}} \left| \mathbb{P}\left[\hat{S}_n \le q \right] - \mathbb{P}\left[S \le q \right] \right| \to 0,$$

hence

$$\sup_{q\in\mathbb{R}} \left| \mathbb{P}\left[\tilde{S}_n > q \right] - \mathbb{P}\left[S > q \right] \right| \to 0, \sup_{q\in\mathbb{R}} \left| \mathbb{P}\left[\hat{S}_n > q \right] - \mathbb{P}\left[S > q \right] \right| \to 0,$$

so Theorem 2 follows immediately by plugging in $q = Q_{1-\alpha}$ and $\mathbb{P}\left[S > Q_{1-\alpha} \right] = \alpha$. □

S.5 Proof of Theorem 3

By Lemma S.3 and Proposition 2, under H_0

$$\left|\sum_{1\leq k<k'\leq\kappa_n}\hat{\lambda}_k\hat{\lambda}_{k'}\chi^2_{kk'}(1)-\sum_{1\leq k<k'\leq\kappa_n}\lambda_k\lambda_{k'}\chi^2_{kk'}(1)\right|$$
$$\leq \kappa_n^2\max_{1\leq k<k'\leq\kappa_n}\chi^2_{kk'}(1)\max_{1\leq k<k'\leq\kappa_n}\left|\hat{\lambda}_k\hat{\lambda}_{k'}-\lambda_k\lambda_{k'}\right|$$
$$\leq \kappa_n^2\times\mathcal{O}_{a.s.}(\log n)\times\max_{1\leq k<k'\leq\kappa_n}\left(\left|\hat{\lambda}_k-\lambda_k\right|\left|\hat{\lambda}_{k'}-\lambda_{k'}\right|+\lambda_k\left|\hat{\lambda}_{k'}-\lambda_{k'}\right|+\lambda_{k'}\left|\hat{\lambda}_k-\lambda_k\right|\right)$$
$$\leq \kappa_n^2\times\mathcal{O}_{a.s.}(\log n)\times\left(\max_{1\leq k\leq\kappa_n}\left|\hat{\lambda}_k-\lambda_k\right|^2+\lambda_1\max_{1\leq k\leq\kappa_n}\left|\hat{\lambda}_k-\lambda_k\right|\right)$$
$$\leq \kappa_n^2\times\mathcal{O}_{a.s.}(\log n)\times\mathcal{O}_{a.s.}\left(n^{-1/2}\log^{1/2}n\right)$$
$$=\mathcal{O}_{a.s.}\left(\kappa_n^2 n^{-1/2}\log^{3/2}n\right).$$

By recalling (2.34) that $\kappa_n^2 n^{-1/2}\log^{3/2}n\to 0$,

$$\left|\sum_{1\leq k<k'\leq\kappa_n}\hat{\lambda}_k\hat{\lambda}_{k'}\chi^2_{kk'}(1)-\sum_{1\leq k<k'\leq\kappa_n}\lambda_k\lambda_{k'}\chi^2_{kk'}(1)\right|=o_{a.s.}(1).$$

Note next that for $1\leq k<k', \mathbb{E}\chi^2_{kk'}(1)=1$, hence by taking $R_{kk',n}=\chi^2_{kk'}(1), M=1$ in Lemma S.5, one obtains that

$$\sum_{1\leq k<k',k'>\kappa_n}\lambda_k\lambda_{k'}\chi^2_{kk'}(1)=o_p(1).$$

Combining the above, one obtains that

$$\left|\bar{S}_n-S\right|=\left|\sum_{1\leq k<k'\leq\kappa_n}\hat{\lambda}_k\hat{\lambda}_{k'}\chi^2_{kk'}(1)-\sum_{1\leq k<k'<\infty}\lambda_k\lambda_{k'}\chi^2_{kk'}(1)\right|$$
$$\leq\left|\sum_{1\leq k<k'\leq\kappa_n}\hat{\lambda}_k\hat{\lambda}_{k'}\chi^2_{kk'}(1)-\sum_{1\leq k<k'\leq\kappa_n}\lambda_k\lambda_{k'}\chi^2_{kk'}(1)\right|+\sum_{1\leq k<k',k'>\kappa_n}\lambda_k\lambda_{k'}\chi^2_{kk'}(1)$$

$$= o_{a.s.}(1) + o_p(1) = o_p(1).$$

Since the distribution function F_S of S is strictly increasing and continuous and $\bar{S}_n - S = o_p(1)$, Lemma 11.2.1(ii) of Lehmann et al. (2005) ensures that the $100(1-\alpha)$-th percentile $\hat{Q}_{1-\alpha}$ of $\bar{S}_n$ converges to that of S, i.e., $\left|\hat{Q}_{1-\alpha} - Q_{1-\alpha}\right| = o_p(1)$. Applying Slutsky's Theorem to

$$\hat{S}_n \xrightarrow{D} S, Q_{1-\alpha} - \hat{Q}_{1-\alpha} \to_p 0,$$

one obtains that

$$\hat{S}_n + Q_{1-\alpha} - \hat{Q}_{1-\alpha} \xrightarrow{D} S.$$

This means that as $n \to \infty$,

$$\mathbb{P}\left[\hat{S}_n + Q_{1-\alpha} - \hat{Q}_{1-\alpha} \le Q_{1-\alpha}\right] \to 1 - \alpha$$

or equivalently

$$\mathbb{P}\left[\hat{S}_n > \hat{Q}_{1-\alpha}\right] \to \alpha.$$

Likewise, $\mathbb{P}\left(\tilde{S}_n > \hat{Q}_{1-\alpha}\right) \to \alpha$. This has completed the proof. □

S.6 Proof of Theorem 4

Under H_1 in (2.13), there exist $k_1 < k_2 \in \mathbb{N}_+$ such that

$$C_{k_1 k_2} = \int G(x, x')\, \psi_{0,k_1}(x)\, \psi_{0,k_2}(x')\, dx dx' \neq 0. \tag{S.44}$$

Applying Hilbert space Central Limit Theorem of (2.28), one concludes that $n^{1/2}\left(Z_{k_1k_2} - C_{k_1k_2}\right)$ converges weakly to a Gaussian distribution of mean zero and finite variance, thus

$$Z_{k_1k_2} = C_{k_1k_2} + \mathcal{O}_p\left(n^{-1/2}\right), Z_{k_1k_2}^2 = C_{k_1k_2}^2 + \mathcal{O}_p\left(n^{-1/2}\right). \tag{S.45}$$

According to (2.20), $\tilde{S}_n = n \sum_{1\le k<k'<\infty} Z_{kk'}^2 \ge nZ_{k_1k_2}^2$, so applying (S.45), as $n \to \infty$

$$\mathbb{P}\left[\tilde{S}_n \ge Q_{1-\alpha}\right] \ge \mathbb{P}\left[nZ_{k_1k_2}^2 \ge Q_{1-\alpha}\right] = \mathbb{P}\left[Z_{k_1k_2}^2 \ge Q_{1-\alpha}/n\right]$$

$$= \mathbb{P}\left[C_{k_1k_2}^2 \ge Q_{1-\alpha}/n + \mathcal{O}_p\left(n^{-1/2}\right)\right] = \mathbb{P}\left[C_{k_1k_2}^2 \ge \mathcal{O}_p\left(n^{-1/2}\right)\right] \to 1,$$

because $C_{k_1k_2}^2 > 0$ according to (S.44).

Putting together (S.24) in Proposition 2 and (S.45), one obtains

$$\hat{Z}_{k_1k_2} = C_{k_1k_2} + \mathcal{O}_p\left(n^{-1/2}\right), \hat{Z}_{k_1k_2}^2 = C_{k_1k_2}^2 + \mathcal{O}_p\left(n^{-1/2}\right). \tag{S.46}$$

Since $\kappa_n \to \infty$ by (2.34), so for large enough n, $\kappa_n > \max(k_1, k_2)$, and according to (2.33) $\hat{S}_n = n \sum_{1\le k<k'\le\kappa_n} \hat{Z}_{kk'}^2$, so $\hat{S}_n \ge n\hat{Z}_{k_1k_2}^2$. Consequently, applying (S.46), as $n \to \infty$

$$\mathbb{P}\left[\hat{S}_n \ge Q_{1-\alpha}\right] \ge \mathbb{P}\left[n\hat{Z}_{k_1k_2}^2 \ge Q_{1-\alpha}\right] = \mathbb{P}\left[\hat{Z}_{k_1k_2}^2 \ge Q_{1-\alpha}/n\right]$$

$$= \mathbb{P}\left[C_{k_1k_2}^2 \ge Q_{1-\alpha}/n + \mathcal{O}_p\left(n^{-1/2}\right)\right] = \mathbb{P}\left[C_{k_1k_2}^2 \ge \mathcal{O}_p\left(n^{-1/2}\right)\right] \to 1,$$

again making use of $C_{k_1k_2}^2 > 0$ according to (S.44).

Making use of (S.23), one obtains

$$\sum_{1\leq k<k'\leq \kappa_n} \hat{\lambda}_k \hat{\lambda}_{k'} \chi^2_{kk'}(1) \leq \Lambda_n^2 \sum_{1\leq k<k'\leq \kappa_n} \chi^2_{kk'}(1) = \Lambda_n^2 \chi^2\left(\frac{\kappa_n(\kappa_n-1)}{2}\right).$$

Note that for the standard χ^2 distribution with degree of freedom $\frac{\kappa_n(\kappa_n-1)}{2} \to \infty$, its $(1-\alpha)$-th quantile is asymptotically

$$\frac{\kappa_n(\kappa_n-1)}{2} + \kappa_n^{1/2}(\kappa_n-1)^{1/2} z_{1-\alpha/2} + o(\kappa_n) \leq 3\kappa_n^2$$

for large enough n. Thus for large enough n, $\hat{Q}_{1-\alpha} \leq 3\kappa_n^2\Lambda_n^2$, which implies that for large enough n,

$$\mathbb{P}\left[\hat{S}_n \geq \hat{Q}_{1-\alpha}\right] \geq \mathbb{P}\left[n\hat{Z}^2_{k_1k_2} \geq 3\kappa_n^2\Lambda_n^2\right] = \mathbb{P}\left[\hat{Z}^2_{k_1k_2} \geq 3\kappa_n^2\Lambda_n^2/n\right]$$

$$= \mathbb{P}\left[C^2_{k_1k_2} \geq 3\kappa_n^2\Lambda_n^2/n + \mathcal{O}_p\left(n^{-1/2}\right)\right] = \mathbb{P}\left[C^2_{k_1k_2} \geq \mathcal{O}_p\left(n^{-1/2}\right)\right] \to 1,$$

again making use of $C^2_{k_1k_2} > 0$ according to (S.44). The arguments for $\mathbb{P}\left[\tilde{S}_n \geq \hat{Q}_{1-\alpha}\right]$ is similar.

The proof is completed. □

S.7 Proof of Theorem 5

For B-spline trajectory estimates, (S.39) in Lemma S.12 provides that

$$\max_{1\leq i\leq n}\left\|\hat{\xi}_i - \xi_i + n^{-1}\sum_{i'=1}^{n}\xi_{i'}\right\|_\infty = \mathcal{O}_{a.s.}\left(J_s^{-p^*}(n\log n)^{2/r_1} + N^{-1/2}J_s^{1/2}\log^{1/2}N + J_sN^{\beta_2-1}\right),$$

hence Assumption (B2) is verified if one shows that

$$\kappa_n^2 n^{1/2}\left\{J_s^{-p^*}(n\log n)^{2/r_1} + N^{-1/2}J_s^{1/2}\log^{1/2}N + J_sN^{\beta_2-1}\right\}\log^{1/2}n \to 0,$$

for some $\{\kappa_n\}_{n=1}^{\infty}$ that satisfy (2.34). One needs to show only that

$$\begin{aligned} n^{1/2}\log^{1/2} n \times J_s^{-p^*}(n\log n)^{2/r_1} &\rightarrow 0, \\ n^{1/2}\log^{1/2} n \times N^{-1/2}J_s^{1/2}\log^{1/2} N &\rightarrow 0, \\ n^{1/2}\log^{1/2} n \times J_s N^{\beta_2-1} &\rightarrow 0, \end{aligned}$$

which follow from Assumptions (C3)-(C5), with all constraints (2.42), (2.43), (2.44), (2.45) on $\theta, p^*, \beta_2, r_1, \gamma$ included.

The proof is completed. □

References

Bosq, D. (1998). *Nonparametric Statistics for Stochastic Processes.* Springer-Verlag, New York.

Bosq, D. (2000). *Linear Processes in Function Spaces: Theory and Applications.* Springer-Verlag, New York.

Cao, G., Yang L. and Todem, D. (2012). Simultaneous inference for the mean function based on dense functional data. *Journal of Nonparametric Statistics* **24**, 359-377.

Csőrgő, M. and Révész, P. (1981). *Strong Approximations in Probability and Statistics.* Academic Press, New York-London.

Christoph, G., Prokhorov, Y. V. and Ulyanov, V. V. (1996). On distribution of quadratic forms in Gaussian random variables. *Theory of Probability & Its Applications* **40**, 250-260.

DeVore, R. and Lorentz, G. (1993). *Constructive Approximation: Polynomials and Splines*

Approximation. Springer-Verlag, Berlin.

Hoeffding, W. (1964). On a theorem of VM Zolotarev. *Theory of Probability & Its Applications* **9**, 89-91.

Lehmann, E. L.,Romano, J. P. and Casella, G. (2005). *Testing Statistical Hypotheses*. Springer-Verlag, New York.

Wang, J., Cao, G., Wang, L. and Yang, L. (2020). Simultaneous confidence band for stationary covariance function of dense functional data. *Journal of Multivariate Analysis* **176**, 104584.

Zolotarev, V. M. (1961). Concerning a certain probability problem. *Theory of Probability & Its Applications* **6**, 201-204.