Supplement to “Oracally Efficient Estimation and Consistent Model Selection for ARMA Time Series with Trend”

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This Supplement gives the complete proof of Theorem 4 that the B-spline estimator \( \hat{g}_m(\omega) \) in (11) satisfies Assumption (c). To this end, the estimator \( \hat{g}_m(\omega) \) is decomposed as

\[
\hat{g}_m(\omega) = \bar{g}_m(\omega) + \tilde{x}_{n,m}
\]

with noise term \( \tilde{x}_{n,m} \) and signal term \( \bar{g}_m(\omega) \) defined as follows:

\[
\tilde{x}_{n,m} = c^T_m(\omega) \left( \frac{1}{n} C^T_m C_m \right)^{-1} \left( \frac{1}{n} C^T_m x \right),
\]

\[
\bar{g}_m(\omega) = c^T_m(\omega) \left( \frac{1}{n} C^T_m C_m \right)^{-1} \left( \frac{1}{n} C^T_m g \right),
\]

with design matrix \( C_m \) in (8). Denote the set of indices \( i \in \{1, \ldots, n\} \) for which \( c_{j,m}(\omega_i) \neq 0 \) as \( I_{j,m} = \{1 \leq i \leq n : c_{j,m}(\omega_i) \neq 0\} \), and the number of elements in \( I_{j,m} \) as \( n_{j,m} \); for any symmetric real matrix \( A \), denote by \( \lambda_{\max}(A) \) the largest eigenvalue of \( A \). The next lemma sums up some basic facts related to B-splines.

**Lemma S.1.** Under Assumptions (a)-(b), (c1)-(c2), as \( n \rightarrow \infty \),

1. (1)
   \[
   \max_{1 \leq t \leq n} E \left( \tilde{x}_{t,j,m}^2 \right) = O \left\{ (n^{-1}N) = O \left\{ n^{-1}h^{-1} \right\} \right.;
   \]
   \[= O \left\{ \frac{1}{n} \right\} \] (S.1)
   \[ \sup_{\omega \in [0,1]} |g(\omega) - \bar{g}_m(\omega)| = O \left( N^{-(m'+\nu)} \right) = O \left( h^{m'+\nu} \right); \] (S.2)

2. (2) there exist constants \( C_\infty, C_\lambda \in (0, +\infty) \) such that

   \[
   \|C_m\|_\infty = \max_{-m+1 \leq j \leq N} \|c_{j,m}\|_\infty \leq C_\infty h^{-1/2}, \quad \lambda_{\max} \left( n^{-1} C^T_m C_m \right)^{-1} \leq C_\lambda; \] (S.3)

3. (3) for each \( 1 - m \leq j \leq N \), the set \( I_{j,m} \) consists of \( n_{j,m} \) consecutive integers from \( \{1, \ldots, n\} \) and that

   \[
   \max_{-m+1 \leq j \leq N} n_{j,m} \leq \lfloor nh \rfloor + 1 \] (S.4)
The “Good Condition” of B-spline basis in Theorem 5.4.2 of DeVore and Lorentz (1993) implies the existence of positive constants $c_0 < C_0$ such that

$$c_0 h \leq \left\| \sum_{j=1}^N b_{j,m}(\omega) \beta_{j,m} \right\|_2 \left( \sum_{j=1}^N \beta_{j,m}^2 \right)^{-1/2} \leq C_0 h, \quad \forall \{\beta_{j,m}\}_{j=1-m}^N \in \mathbb{R}^{N+m}. \quad (S.5)$$

Consequently $c_0 h \leq \|b_{j,m}\|_2 \leq C_0 h$. By the “Partition of Unity” property of B-spline basis (de Boor 2001, page 96), $0 \leq b_{j,m}(\omega) \leq 1$ so $c_{j,m}(\omega) \leq \|b_{j,m}\|_2^{-1} \leq c_0^{-1} h^{-1}$, and the bound in (S.3) on $\|C_m\|_\infty$ follows. The bound on $\lambda_{\text{max}} \left( n^{-1} C_m^T C_m \right)^{-1}$ in (S.3) follows also from (S.5).

The “Partition of Unity” property of B-spline basis (de Boor 2001, page 96) implies that the support of $c_{j,m}$ consists of at most $m$ consecutive integers according to Lemma S.1 (3), (A.1) and (S.4) imply the following

$$\sum_{c_{j,m}(\omega) \in I_{j,m}} \rho_{\xi}^{[k-l]} = \sum_{k,l \in I_{j,m}} \rho_{\xi}^{[k-l]} \leq n_{j,m} \left( 1 - \rho_{\xi} \right)^{-1} \leq ([nh] + m) \left( 1 - \rho_{\xi} \right)^{-1}. \quad (S.6)$$

The above inequality, the $(C_{\xi}, \rho_{\xi})$-exponential correlatedness of $\xi$, and $\min(nh, N/m) > 1$ lead to

$$\text{E}\|C_m^T \xi\|^2 = \sum_{j=-m+1}^N \text{E}(C_{j,m}^T \xi)^2 = \sum_{j=-m+1}^N \sum_{k=1}^n \sum_{l=1}^n c_{j,m}(\omega_k) c_{j,m}(\omega_l) \text{E}(\xi_k \xi_l) \leq \|C_m\|_\infty^2 \sum_{j=-m+1}^N \sum_{c_{j,m}(\omega_k) \in I_{j,m}(\omega_l)} C_{\xi}^{[k-l]} \leq \|C_m\|_\infty^2 C_{\xi} \sum_{j=-m+1}^N ([nh] + m) \left( 1 - \rho_{\xi} \right)^{-1} \left( N + m \right) ([nh] + 1) m \leq \|C_m\|_\infty^2 C_{\xi} \left( 1 - \rho_{\xi} \right)^{-1} \left( 2N \right) (2 \times nh) m = 4m \|C_m\|_\infty^2 C_{\xi} \left( 1 - \rho_{\xi} \right)^{-1} Nnh.$$ 

Making use of (S.3) in Lemma S.1 (2), one obtains that

$$\text{E}\|C_m^T \xi\|^2 \leq 4mC_{\xi}^2 h^{-1} C_{\xi} \left( 1 - \rho_{\xi} \right)^{-1} Nnh = 4mC_{\xi}^2 C_{\xi} \left( 1 - \rho_{\xi} \right)^{-1} nN.$$ 

The proof is complete. \qed
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Lemma S.3. Under Assumptions (a)-(b), (c1)-(c2), for any $1 \leq k \leq p + q$, as $n \to \infty$,

\[
T_{1n} = n^{-1} \sum_{t=p+1}^{n} \left( \sum_{j=0}^{t-1} \theta \pi_{0j} \pi_{t-j} \right) \left( \sum_{j=0}^{t-1} \pi_{0j} \{ g(\omega_{t-j}) - \tilde{g}_m(\omega_{t-j}) \} \right) = o_p(n^{-1/2}),
\]

\[
T_{2n} = n^{-1} \sum_{t=p+1}^{n} \left( \sum_{j=0}^{t-1} \pi_{0j} x_t \pi_{t-j} \right) \left( \sum_{j=0}^{t-1} \theta \pi_{0j} \pi_{t-j} \right) \left( \sum_{j=0}^{t-1} \pi_{0j} \{ g(\omega_{t-j}) - \tilde{g}_m(\omega_{t-j}) \} \right) = o_p(n^{-1/2}),
\]

\[
T_{3n} = n^{-1} \sum_{t=p+1}^{n} \left( \sum_{j=0}^{t-1} \pi_{0j} x_t \pi_{t-j} \right) \left( \sum_{j=0}^{t-1} \theta \pi_{0j} \pi_{t-j} \right) \left( \sum_{j=0}^{t-1} \pi_{0j} \{ g(\omega_{t-j}) - \tilde{g}_m(\omega_{t-j}) \} \right) = o_p(n^{-1/2}),
\]

\[
T_{4n} = n^{-1} \sum_{t=p+1}^{n} \left( \sum_{j=0}^{t-1} \theta \pi_{0j} \pi_{t-j} \right) \left( \sum_{j=0}^{t-1} \pi_{0j} \{ g(\omega_{t-j}) - \tilde{g}_m(\omega_{t-j}) \} \right) = o_p(n^{-1/2}).
\]

Proof. We provide detailed proofs only for $T_{1n}$ and $T_{3n}$ as the proofs for $T_{2n}$ and $T_{4n}$ are similar.

We begin by noting that from the inequality (27) of Yao and Brockwell (2006), $\{ \theta \pi_{0j} / \pi_{t-j} \}_{j=0}^{n-1}$ is $(C, s)$-exponentially bounded for some $C > 0, s \in (0, 1)$, while $\pi_{0j} \}_{j=0}^{n-1}$ is $(C_\pi, \rho_\pi)$-exponentially bounded by (15).

Now equation (3.3.10) of Brockwell and Davis (1991) ensures that $x = (x_1, \ldots, x_n)^T$ is $(C_x, \rho_x)$-exponentially correlated for some $C_x > 0, \rho_x \in (0, 1)$. Lemma A.3 then implies that the sequence $\{ \sum_{j=0}^{t-1} (\theta \pi_{0j} / \pi_{t-j}) \}_{t=p+1}^{n}$ is $(C_\pi, \rho_\pi)$-exponentially correlated for some $C_\pi > 0, \rho_\pi \in (0, 1)$. Meanwhile, Lemma A.1 provides that

\[
\max_{p+1 \leq t \leq n} \left| \sum_{j=0}^{t-1} \pi_{0j} \{ g(\omega_{t-j}) - \tilde{g}_m(\omega_{t-j}) \} \right| \leq C_\pi (1 - \rho_\pi)^{-1} \max_{1 \leq t \leq n} |g(\omega_t) - \tilde{g}_m(\omega_t)| = O \left( h^{m' + \nu} \right),
\]

where the last inequality uses (S.2). Now applying Lemma A.2 to the $(C_\pi, \rho_\pi)$-exponentially correlated $\{ \sum_{j=0}^{t-1} (\theta \pi_{0j} / \pi_{t-j}) \}_{t=p+1}^{n}$ and the deterministic sequence $\{ \sum_{j=0}^{t-1} \pi_{0j} \{ g(\omega_{t-j}) - \tilde{g}_m(\omega_{t-j}) \} \}_{t=p+1}^{n}$ with uniform bound $O \left( h^{m' + \nu} \right)$, one obtains that

\[
T_{1n} = n^{-1} \left| \sum_{t=p+1}^{n} \left( \sum_{j=0}^{t-1} \frac{\theta \pi_{0j} x_t \pi_{t-j}}{\pi_{t-j}} \right) \left( \sum_{j=0}^{t-1} \pi_{0j} \{ g(\omega_{t-j}) - \tilde{g}_m(\omega_{t-j}) \} \right) \right| = O_p \left( n^{-1} \times n^{1/2} h^{m' + \nu} \right) = O_p \left( n^{-1/2} h^{m' + \nu} \right) = o_p \left( n^{-1/2} \right),
\]

as $m' + \nu > 1/2, h^{m' + \nu} \to 0$, hence the proof for term $T_{1n}$ is completed.

Define next $(n - p) \times n$ matrices

\[
W_1 = \begin{pmatrix}
\pi_{0,0} & \pi_{0,0-p} & \cdots & \pi_{0,1} & 1 & 0 & \cdots & 0 & 0 \\
\pi_{0,p+1} & \pi_{0,p} & \cdots & \pi_{0,2} & \pi_{0,1} & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\pi_{0,n-1} & \pi_{0,n-2} & \cdots & \pi_{0,n-p} & \pi_{0,n-p-1} & \pi_{0,n-p-2} & \cdots & \pi_{0,1} & 1
\end{pmatrix},
\]
Define $\eta = W_2^T W_1 x$. Since $(\partial \pi_{0j}/\partial \alpha_k)_{j=0}^{n-1}$ is $(C, s)$-exponentially bounded and $(\pi_{0j})_{j=0}^{n-1}$ is $(C_\pi, \rho_\pi)$-exponentially bounded, Lemma A.3 applied twice imply that $\eta$ is $(C_\eta, \rho_\eta)$-exponentially correlated for some $C_\eta > 0, \rho_\eta \in (0, 1)$. According to Lemma S.2, $E\|C_m^T \eta\|^2 \leq 4mC^2_\infty C_\eta (1 - \rho_\eta)^{-1} nN$, when $\min(\eta, N/m) > 1$, hence $\|C_m \eta\| = O_p \left( (nN)^{1/2} \right)$. Likewise, since $x = (x_1, \cdots, x_n)^T$ is $(C_x, \rho_x)$-exponentially correlated, $E\|C^T_m x\|^2 \leq 4mC^2_\infty C_x (1 - \rho_x)^{-1} nN$, when $\min(\eta, N/m) > 1$, hence $\|C^T_m x\| = O_p \left( (nN)^{1/2} \right)$. Since $\lambda_{\max} (n^{-1} C_m^T C_m)^{-1} \leq C_\lambda$ by (S.3), the term $T_3a$ is bounded by

$$n^{-1} |x^T W_1^T W_2 \tilde{x}| = n^{-2} |\eta^T C_m (n^{-1} C^T_m C_m)^{-1} C^T_m x| \leq n^{-2} \|C^T_m \eta\| \lambda_{\max} (n^{-1} C^T_m C_m)^{-1} \|C^T_m x\| \
\leq C_{\lambda} n^{-2} \times O_p \left( (nN)^{1/2} \right) \times O_p \left( (nN)^{1/2} \right) = O_p (n^{-1/2}),$$

where the last inequality is due to $N \ll n^{1/2}$ in Assumption (c2). The proof is complete.

**Proof of Theorem 4.** To show (12), note that

$$\{g(\omega_t) - \tilde{g}_m(\omega_t)\}^2 = \{g(\omega_t) - \tilde{g}_m(\omega_t) - \tilde{x}_{t-j}\}^2 \leq 2 \{g(\omega_t) - \tilde{g}_m(\omega_t)\}^2 + 2 \tilde{x}_{t-j}^2$$

hence (S.2) and (S.1) provide that

$$\max_{1 \leq t \leq n} E \{g(\omega_t) - \tilde{g}_m(\omega_t)\}^2 \leq 2 \max_{1 \leq t \leq n} \{g(\omega_t) - \tilde{g}_m(\omega_t)\}^2 + 2 \max_{1 \leq t \leq n} E \tilde{x}_{t-j}^2 \leq \left\{ \sup_{\omega \in [0, 1]} |g(\omega) - \tilde{g}_m(\omega)| \right\}^2 + O \left\{ (nh)^{-1} \right\} = O \left( N^{-2(m'+\nu)} + n^{-1} N \right) = o \left( n^{-1/2} \right),$$

the last inequality following from Assumption (c2) that $n^{1/4(m'+\nu)} \ll N \ll n^{1/2}$ which implies that $N^{-2(m'+\nu)} \ll n^{-1/2}$ and that $n^{-1} N \ll n^{-1/2}$. Since (12) follows from (S.6), while (13) and (14) follow from Lemma S.3, the proof is complete for Theorem 4. \hfill \Box