Supplement to “Oracally Efficient Estimation and Consistent Model Selection for ARMA Time Series with Trend”

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This Supplement gives the complete proof of Theorem 4 that the B-spline estimator $\hat{g}_m(\omega)$ in (11) satisfies Assumption (c). To this end, the estimator $\hat{g}_m(\omega)$ is decomposed as

$$\hat{g}_m(\omega) = \tilde{g}_m(\omega) + \tilde{x}_{n\omega,m}$$

with noise term $\tilde{x}_{n\omega,m}$ and signal term $\tilde{g}_m(\omega)$ defined as follows:

$$\tilde{x}_{n\omega,m} = \mathbf{c}_m^T(\omega)\left(\frac{1}{n}\mathbf{C}_m^T\mathbf{C}_m\right)^{-1}\left(\frac{1}{n}\mathbf{C}_m^T\mathbf{x}\right),$$
$$\tilde{g}_m(\omega) = \mathbf{c}_m^T(\omega)\left(\frac{1}{n}\mathbf{C}_m^T\mathbf{C}_m\right)^{-1}\left(\frac{1}{n}\mathbf{C}_m^T\mathbf{g}\right),$$

with design matrix $\mathbf{C}_m$ in (8). Denote the set of indices $i \in \{1,\cdots,n\}$ for which $c_{j,m}(\omega_i) \neq 0$ as $I_{j,m}$ $= \{1 \leq i \leq n : c_{j,m}(\omega_i) \neq 0\}$, and the number of elements in $I_{j,m}$ as $n_{j,m}$; for any symmetric real matrix $\mathbf{A}$, denote by $\lambda_{\max}(\mathbf{A})$ the largest eigenvalue of $\mathbf{A}$. The next lemma sums up some basic facts related to B-splines.

LEMMA S.1. *Under Assumptions (a)-(b), (c1)-(c2), as $n \to \infty$,*
(1)

$$\max_{1\leq t\leq n} E\left(\tilde{x}_{t,m}^2\right) = O\left\{(n^{-1}N\right\} = O\left\{n^{-1}h^{-1}\right\}, \tag{S.1}$$

$$\sup_{\omega\in[0,1]} \left|g(\omega) - \tilde{g}_m(\omega)\right| = O\left(N^{-(m'+\nu)}\right) = O\left(h^{m'+\nu}\right); \tag{S.2}$$

(2) there exist constants $C_\infty, C_\lambda \in (0,+\infty)$ such that

$$\|\mathbf{C}_m\|_\infty = \max_{-m+1\leq j\leq N} \|\mathbf{c}_{j,m}\|_\infty \leq C_\infty h^{-1/2}, \quad \lambda_{\max}\left(n^{-1}\mathbf{C}_m^T\mathbf{C}_m\right)^{-1} \leq C_\lambda; \tag{S.3}$$

(3) for each $1-m \leq j \leq N$, the set $I_{j,m}$ consists of $n_{j,m}$ consecutive integers from $\{1,\cdots,n\}$ and that

$$\max_{-m+1\leq j\leq N} n_{j,m} \leq ([nh]+1)\,m. \tag{S.4}$$

Proof. Lemma 4.5 of Shao and Yang (2012), Theorem 5.1 of Huang (2003) and equation (8) page 149 of de Boor (2001) prove (S.1) and (S.2).

The "Good Condition" of B-spline basis in Theorem 5.4.2 of DeVore and Lorentz (1993) implies the existence of positive constants $c_0 < C_0$ such that

$$c_0 h \leq \left\| \sum_{j=1-m}^{N} b_{j,m}(\omega)\beta_{j,m} \right\|_2 \left(\sum_{j=1-m}^{N} \beta_{j,m}^2 \right)^{-1/2} \leq C_0 h, \quad \forall \{\beta_{j,m}\}_{j=1-m}^{N} \in \mathbb{R}^{N+m}. \tag{S.5}$$

Consequently $c_0 h \leq \|b_{j,m}\|_2 \leq C_0 h$. By the "Partition of Unity" property of B-spline basis (de Boor 2001, page 96), $0 \leq b_{j,m}(\omega) \leq 1$ so $c_{j,m}(\omega) \leq \|b_{j,m}\|_2^{-1} \leq c_0^{-1} h^{-1}$, and the bound in (S.3) on $\|\mathbf{C}_m\|_\infty$ follows. The bound on $\lambda_{\max}\left(n^{-1}\mathbf{C}_m^T \mathbf{C}_m\right)^{-1}$ in (S.3) follows also from (S.5).

The "Partition of Unity" property of B-spline basis (de Boor 2001, page 96) implies that the support of $c_{j,m}$ consists of at most m consecutive intervals each of length h, the consecutiveness of set $I_{j,m}$ and the bound in (S.4) on $n_{j,m}$ then follow. □

Lemma S.2. *If $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_n)^T$ is a (C_ξ, ρ_ξ)-exponentially correlated random sequence, then for $\mathbf{C}_m$ given in (8), $E\|\mathbf{C}_m^T \boldsymbol{\xi}\|^2 \leq 4mC_\infty^2 C_\xi (1-\rho_\xi)^{-1} nN$, when $\min(nh, N/m) > 1$, with constant C_∞ as defined in Lemma* S.1.

Proof. Since $I_{j,m}$ consists of consecutive integers according to Lemma S.1 (3), (A.1) and (S.4) imply the following

$$\sum_{c_{j,m}(\omega_k)c_{j,m}(\omega_l)>0} \rho_\xi^{|k-l|} = \sum_{k,l \in I_{j,m}} \rho_\xi^{|k-l|} \leq n_{j,m}(1-\rho_\xi)^{-1} \leq ([nh]+m)(1-\rho_\xi)^{-1}.$$

The above inequality, the (C_ξ, ρ_ξ)-exponential correlatedness of $\boldsymbol{\xi}$, and $\min(nh, N/m) > 1$ lead to

$$\begin{aligned}
\mathrm{E}\|\mathbf{C}_m^T \boldsymbol{\xi}\|^2 &= \sum_{j=-m+1}^{N} \mathrm{E}\left(\mathbf{c}_{j,m}^T \boldsymbol{\xi}\right)^2 = \sum_{j=-m+1}^{N} \sum_{k=1}^{n} \sum_{l=1}^{n} c_{j,m}(\omega_k) c_{j,m}(\omega_l) \mathrm{E}(\xi_k \xi_l) \\
&\leq \|\mathbf{C}_m\|_\infty^2 \sum_{j=-m+1}^{N} \sum_{k=1}^{n} \sum_{l=1}^{n} |\mathrm{E}(\xi_k \xi_l)| \leq \|\mathbf{C}_m\|_\infty^2 \sum_{j=-m+1}^{N} \sum_{c_{j,m}(\omega_k)c_{j,m}(\omega_l)>0} C_\xi \rho_\xi^{|k-l|} \\
&\leq \|\mathbf{C}_m\|_\infty^2 C_\xi \sum_{j=-m+1}^{N} ([nh]+1) m (1-\rho_\xi)^{-1} = \|\mathbf{C}_m\|_\infty^2 C_\xi (1-\rho_\xi)^{-1} (N+m)([nh]+1) m \\
&\leq \|\mathbf{C}_m\|_\infty^2 C_\xi (1-\rho_\xi)^{-1} (2N)(2 \times nh) m = 4m \|\mathbf{C}_m\|_\infty^2 C_\xi (1-\rho_\xi)^{-1} Nnh.
\end{aligned}$$

Making use of (S.3) in Lemma S.1 (2), one obtains that

$$\mathrm{E}\|\mathbf{C}_m^T \boldsymbol{\xi}\|^2 \leq 4mC_\infty^2 h^{-1} C_\xi (1-\rho_\xi)^{-1} Nnh = 4mC_\infty^2 C_\xi (1-\rho_\xi)^{-1} nN.$$

The proof is complete. □

LEMMA S.3. *Under Assumptions (a)-(b), (c1)-(c2), for any $1 \le k \le p+q$, as $n \to \infty$,*

$$\begin{aligned}
T_{1n} &= n^{-1}\left|\sum_{t=p+1}^{n}\left(\sum_{j=0}^{t-1}\frac{\partial \pi_{0j}}{\partial \alpha_k}x_{t-j}\right)\left[\sum_{j=0}^{t-1}\pi_{0j}\left\{g(\omega_{t-j})-\tilde{g}_m(\omega_{t-j})\right\}\right]\right| = o_p(n^{-1/2}),\\
T_{2n} &= n^{-1}\left|\sum_{t=p+1}^{n}\left(\sum_{j=0}^{t-1}\pi_{0j}x_{t-j}\right)\left[\sum_{j=0}^{t-1}\frac{\partial \pi_{0j}}{\partial \alpha_k}\left\{g(\omega_{t-j})-\tilde{g}_m(\omega_{t-j})\right\}\right]\right| = o_p(n^{-1/2}),\\
T_{3n} &= n^{-1}\left|\sum_{t=p+1}^{n}\left(\sum_{j=0}^{t-1}\pi_{0j}x_{t-j}\right)\left(\sum_{j=0}^{t-1}\frac{\partial \pi_{0j}}{\partial \alpha_k}\tilde{x}_{t-j,m}\right)\right| = o_p(n^{-1/2}),\\
T_{4n} &= n^{-1}\left|\sum_{t=p+1}^{n}\left(\sum_{j=0}^{t-1}\frac{\partial \pi_{0j}}{\partial \alpha_k}x_{t-j}\right)\left(\sum_{j=0}^{t-1}\pi_{0j}\tilde{x}_{t-j,m}\right)\right| = o_p(n^{-1/2}).
\end{aligned}$$

Proof. We provide detailed proofs only for T_{1n} and T_{3n} as the proofs for T_{2n} and T_{4n} are similar.

We begin by noting that from the inequality (27) of Yao and Brockwell (2006), $\{\partial\pi_{0j}/\partial\alpha_k\}_{j=0}^{n-1}$ is (C,s)-exponentially bounded for some $C > 0, s \in (0,1)$, while $\{\pi_{0j}\}_{j=0}^{n-1}$ is (C_π, ρ_π)-exponentially bounded by (15).

Now equation (3.3.10) of Brockwell and Davis (1991) ensures that $\mathbf{x} = (x_1, \cdots, x_n)^T$ is (C_x, ρ_x)-exponentially correlated for some $C_x > 0, \rho_x \in (0,1)$. Lemma A.3 then implies that the sequence $\left\{\sum_{j=0}^{t-1}(\partial\pi_{0j}/\partial\alpha_k)x_{t-j}\right\}_{t=p+1}^{n}$ is (C_ζ, ρ_ζ)-exponentially correlated for some $C_\zeta > 0, \rho_\zeta \in (0,1)$. Meanwhile, Lemma A.1 provides that

$$\max_{p+1\le t\le n}\left|\sum_{j=0}^{t-1}\pi_{0j}\left\{g(\omega_{t-j})-\tilde{g}_m(\omega_{t-j})\right\}\right| \le C_\pi(1-\rho_\pi)^{-1}\max_{1\le t\le n}\left|g(\omega_t)-\tilde{g}_m(\omega_t)\right| = O\left(h^{m'+\nu}\right),$$

where the last inequality uses (S.2). Now applying Lemma A.2 to the (C_ζ, ρ_ζ)-exponentially correlated $\left\{\sum_{j=0}^{t-1}(\partial\pi_{0j}/\partial\alpha_k)x_{t-j}\right\}_{t=p+1}^{n}$ and the deterministic sequence $\left\{\sum_{j=0}^{t-1}\pi_{0j}\left\{g(\omega_{t-j})-\tilde{g}_m(\omega_{t-j})\right\}\right\}_{t=p+1}^{n}$ with uniform bound $O\left(h^{m'+\nu}\right)$, one obtains that

$$\begin{aligned}
T_{1n} &= n^{-1}\left|\sum_{t=p+1}^{n}\left(\sum_{j=0}^{t-1}\frac{\partial \pi_{0j}}{\partial \phi_k}x_{t-j}\right)\left[\sum_{j=0}^{t-1}\pi_{0j}\left\{g(\omega_{t-j})-\tilde{g}_m(\omega_{t-j})\right\}\right]\right|\\
&= O_p\left(n^{-1}\times n^{1/2}h^{m'+\nu}\right) = O_p\left(n^{-1/2}h^{m'+\nu}\right) = o_p\left(n^{-1/2}\right),
\end{aligned}$$

as $m'+\nu > 1/2, h^{m'+\nu} \to 0$, hence the proof for term T_{1n} is completed.

Define next $(n-p)\times n$ matrices

$$\mathbf{W}_1 = \begin{pmatrix}
\pi_{0,p} & \pi_{0,p-1} & \cdots & \pi_{0,1} & 1 & 0 & \cdots & 0 & 0\\
\pi_{0,p+1} & \pi_{0,p} & \cdots & \pi_{0,2} & \pi_{0,1} & 1 & \cdots & 0 & 0\\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots\\
\pi_{0,n-1} & \pi_{0,n-2} & \cdots & \pi_{0,n-p} & \pi_{0,n-p-1} & \pi_{0,n-p-2} & \cdots & \pi_{0,1} & 1
\end{pmatrix},$$

$$\mathbf{W}_2 = \begin{pmatrix} \frac{\partial}{\partial\alpha_k}\pi_{0,p} & \frac{\partial}{\partial\alpha_k}\pi_{0,p-1} & \cdots & \frac{\partial}{\partial\alpha_k}\pi_{0,1} & 0 & 0 & \cdots & 0 & 0 \\ \frac{\partial}{\partial\alpha_k}\pi_{0,p+1} & \frac{\partial}{\partial\alpha_k}\pi_{0,p} & \cdots & \frac{\partial}{\partial\alpha_k}\pi_{0,2} & \frac{\partial}{\partial\alpha_k}\pi_{0,1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{\partial}{\partial\alpha_k}\pi_{0,n-1} & \frac{\partial}{\partial\alpha_k}\pi_{0,n-2} & \cdots & \frac{\partial}{\partial\alpha_k}\pi_{0,n-p} & \frac{\partial}{\partial\alpha_k}\pi_{0,n-p-1} & \frac{\partial}{\partial\alpha_k}\pi_{0,n-p-2} & \cdots & \frac{\partial}{\partial\alpha_k}\pi_{0,1} & 0 \end{pmatrix}.$$

Define $\boldsymbol{\eta} = \mathbf{W}_2^T\mathbf{W}_1\mathbf{x}$. Since $(\partial\pi_{0j}/\partial\alpha_k)_{j=0}^{n-1}$ is (C,s)-exponentially bounded and $(\pi_{0j})_{j=0}^{n-1}$ is (C_π,ρ_π)-exponentially bounded, Lemma A.3 applied twice imply that $\boldsymbol{\eta}$ is (C_η,ρ_η)-exponentially correlated for some $C_\eta > 0, \rho_\eta \in (0,1)$. According to Lemma S.2, $\mathrm{E}\|\mathbf{C}_m^T\boldsymbol{\eta}\|^2 \le 4mC_\infty^2 C_\eta (1-\rho_\eta)^{-1} nN$, when $\min(nh, N/m) > 1$, hence $\|\mathbf{C}_m^T\boldsymbol{\eta}\| = O_p\left((nN)^{1/2}\right)$. Likewise, since $\mathbf{x} = (x_1,\cdots,x_n)^T$ is (C_x,ρ_x)-exponentially correlated, $\mathrm{E}\|\mathbf{C}_m^T\mathbf{x}\|^2 \le 4mC_\infty^2 C_x (1-\rho_x)^{-1} nN$, when $\min(nh, N/m) > 1$, hence $\|\mathbf{C}_m^T\mathbf{x}\| = O_p\left((nN)^{1/2}\right)$. Since $\lambda_{\max}\left(n^{-1}\mathbf{C}_m^T\mathbf{C}_m\right)^{-1} \le C_\lambda$ by (S.3), the term T_{3n} is bounded by

$$\begin{aligned} n^{-1}\left|\mathbf{x}^T\mathbf{W}_1^T\mathbf{W}_2\tilde{\mathbf{x}}\right| &= n^{-2}\left|\boldsymbol{\eta}^T\mathbf{C}_m\left(n^{-1}\mathbf{C}_m^T\mathbf{C}_m\right)^{-1}\mathbf{C}_m^T\mathbf{x}\right| \le n^{-2}\|\mathbf{C}_m^T\boldsymbol{\eta}\|\lambda_{\max}\left(n^{-1}\mathbf{C}_m^T\mathbf{C}_m\right)^{-1}\|\mathbf{C}_m^T\mathbf{x}\| \\ &\le C_\lambda n^{-2}\times O_p\left((nN)^{1/2}\right)\times O_p\left((nN)^{1/2}\right) = O_p\left(n^{-1}N\right) = o_p\left(n^{-1/2}\right), \end{aligned}$$

where the last inequality is due to $N \ll n^{1/2}$ in Assumption (c2). The proof is complete. □

Proof of Theorem 4. To show (12), note that

$$\{g(\omega_t) - \hat{g}_m(\omega_t)\}^2 = \{g(\omega_t) - \tilde{g}_m(\omega_t) - \tilde{x}_{t-j}\}^2 \le 2\{g(\omega_t) - \tilde{g}_m(\omega_t)\}^2 + 2\tilde{x}_{t-j}^2$$

hence (S.2) and (S.1) provide that

$$\begin{aligned} &\max_{1\le t\le n}\mathrm{E}\{g(\omega_t) - \hat{g}_m(\omega_t)\}^2 \le 2\max_{1\le t\le n}\{g(\omega_t) - \tilde{g}_m(\omega_t)\}^2 + 2\max_{1\le t\le n}\mathrm{E}\tilde{x}_{t-j}^2 \\ \le\ &\left\{\sup_{\omega\in[0,1]}|g(\omega) - \tilde{g}_m(\omega)|\right\}^2 + O\left\{(nh)^{-1}\right\} = O\left(N^{-2(m'+\nu)} + n^{-1}N\right) = o\left(n^{-1/2}\right), \quad \text{(S.6)} \end{aligned}$$

the last inequality following from Assumption (c2) that $n^{1/4(m'+\nu)} \ll N \ll n^{1/2}$ which implies that $N^{-2(m'+\nu)} \ll n^{-1/2}$ and that $n^{-1}N \ll n^{-1/2}$. Since (12) follows from (S.6), while (13) and (14) follow from Lemma S.3, the proof is complete for Theorem 4. □